

Toward Optimal Time-Space Tradeoffs for Set Reconciliation

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Set reconciliation, where two parties each holding a large set of elements aim to identify their set difference, is a fundamental task in many areas. There are two important metrics in this problem: time (computation cost) and space (communication cost). Most previous work focuses on optimizing one metric at the expense of the other. We present *XYZ-Sketch*, proving that it is possible to achieve near-minimal space and $O(1)$ time updates simultaneously. Specifically, for sufficiently large d , *XYZ-Sketch* reconciles sets with only $(1 + \epsilon)d$ elements for communication, while achieving $O(1)$ insertion time and $O(d \log V)$ decoding time. Here, d and V denote the size of the difference between two sets and the universe size, respectively. We further establish a broad fixed-support canonical model for the problem, showing that, under an open extremality conjecture, our *XYZ-Sketch* is asymptotically optimal within this model. Experiments validate the predicted near-optimal performance of *XYZ-Sketch*. The source code of *XYZ-Sketch* is available at <https://github.com/djwj233/XYZ-Sketch>.

Additional Key Words and Phrases: set reconciliation, streaming sketches, peeling, random hypergraphs

1 Introduction

1.1 The Set Reconciliation Problem

The *set reconciliation* problem asks two parties, Alice and Bob, holding sets $A, B \subseteq \mathcal{U}$, to recover $A \setminus B$ and $B \setminus A$ while communicating as little information as possible. It is a fundamental problem for maintaining consistency among distributed replicas and databases [6, 9, 19], disseminating blocks and transaction sets in blockchain networks [22, 28], and comparing packet observations across network measurement points to detect loss or inconsistency [17, 27].

Let $d := |(A \setminus B) \cup (B \setminus A)|$, $V := |\mathcal{U}|$. We focus on the common regime $d \ll |A \cup B|$. Typically, each party processes its elements into a compact *sketch*. After receiving Alice’s sketch, Bob combines it with his own sketch and decodes the difference. Our main focus is the *streaming* setting, where it is crucial to support $O(1)$ time updates for each arriving element, while decoding is performed only after the stream ends, thus a higher computational overhead is acceptable. Therefore, the key metrics are twofold:

- (1) **Space:** The communication cost, which is measured by the number $\mathfrak{C}$ of transmitted elements in the universe $\mathcal{U}$.¹ A standard information-theoretic lower bound [18] gives $\mathfrak{C} \geq d$ or at least $d \log_2(V/d)$ bits, more precisely.

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¹We use this word-level metric for brevity and convenience, as it closely tracks the actual communication cost of the algorithm. The fully precise bit-level accounting, including all metadata overheads, is provided in the main text.

(2) **Time:** The computational cost of update (and decoding).

Most prior work can be classified into two categories. **1) Algebraic methods** (e.g., CPI[18] and PinSketch[8]). They encode a set as a global polynomial or error-correcting-code syndrome, from which the difference is recovered algebraically. They can approach optimal space, namely $\mathfrak{C} = d + O(1)$, but do not support constant-time updates. **2) Hashing-based methods** (e.g., IBLT[12], Difference Digests[9] and Graphene[22]). They hash each item into $O(1)$ cells and decode by repeatedly peeling cells with one residual item. They support $O(1)$ updates, but require a higher space overhead, typically $\mathfrak{C} \geq 1.23d$. The clear trade-off raises a natural question:

Question 1: *Can one approach the space lower bound while retaining $O(1)$ updates, breaking the current trade-off?*

1.2 Our Solution: XYZ-Sketch

Our answer is **Yes**. In this paper, we propose XYZ-Sketch, which shows that near-minimal space and $O(1)$ -time updates are compatible.

More precisely, for all constants $\varepsilon, \delta > 0$, XYZ-Sketch can reconcile A and B with probability at least $1 - \delta$, use $O(1)$ -time updates and retain $\mathfrak{C} \leq (1 + \varepsilon)d$ for all sufficiently large d . The formal statement appears in Theorem 13.

Table 1. While existing methods focus on either time or space, XYZ-Sketch achieves near-optimality in both.

Algorithm	Space ($\mathfrak{C}$)	Insertion Time	Decoding Time
PinSketch [8]	d	$\Theta(d)$	$\Theta(d^2 + d \log V)$
IBLT [12]	$(1.23 + \varepsilon)d$	$\Theta(1)$	$\Theta(d)$
XYZ-Sketch (Our Work)	$(1 + \varepsilon)d$	$\Theta(1)$	$\Theta(d \log V)$
Separate Lower Bounds for Three Metrics	$\geq d$	$\Omega(1)$	$\Omega(d)$

Our design builds on the IBLT architecture and combines three ingredients, with the key novelty lying in the first two.

1) High-capacity Cells. Let us revisit the decoding process of IBLT. A conventional IBLT maps items to some *cells*. The decoder follows a peeling process: it searches for cells that contain only one differing element (i.e. those in $(A \setminus B) \cup (B \setminus A)$), then recovers the item and removes it from the other cells it touches. Once there is no such cell, the decoding fails immediately.

We introduce *high-capacity cells* that enlarge the capacity of cells from 1 to $\ell > 1$. Specifically, for any fixed constant $\ell > 1$, a high-capacity cell can recover up to ℓ differing elements (e.g., X, Y, Z), hence the name XYZ-Sketch. Our high-capacity cells make the decoding process less prone to failure and thereby allow us to significantly reduce space cost while maintaining the same error rate.

2) Degree-aware RFR. The subproblem within a high-capacity cell is similar to a set reconciliation instance with $d = \ell$. In principle, the inner algorithm can be implemented by existing algebraic methods such as CPI[18] or PinSketch[8]. However, each of them falls short of our requirements in certain aspects.² We therefore develop the *Degree-aware Rational Function Reconstruction* (D-RFR) technique, which enables exact signed recovery within a cell over the native field.

²See related work for details.

3) Spatial Coupling. We leverage the spatial coupling technique from [26], further reducing the number of cells required while preserving $O(1)$ updates. To our knowledge, XYZ-Sketch is the first set-reconciliation data structure to realize ℓ -peeling for $\ell > 1$, thereby providing a concrete application of ℓ -peelability results, and addressing the lack of such applications noted by [26, Section 1.5]. We also introduce the *circular trick* and provide a heuristic analysis of key parameters in spatial coupling, thereby extending its theoretical foundation.

1.3 The Fixed-Support Frontier: Conditional Optimality of XYZ-Sketch

Another natural question is:

Question 2: *Is XYZ-Sketch optimal for the set reconciliation problem?*

Under a natural formalization of the problem, we give a positive answer to this problem:

Our Claim: Under a broad model, XYZ-Sketch is asymptotically optimal, provided that the *Uniform-Support Extremality Conjecture* holds.

To be specific, we formulate a broad model for set reconciliation in the one-pass streaming setting by considering *fixed-support canonical sketches*. Inspired by the cell-probe model [29], we view a persistent sketch as an array of cells. We then require the algorithm to satisfy *fixed-support* and *canonicity* conditions. This broad model imposes no restriction on the decoding strategy. Thus, all existing set-reconciliation sketches known to us satisfy the fixed-support and canonicity conditions.³ We then show that:

THEOREM 1 (MAIN RESULT: CONDITIONAL OPTIMALITY OF XYZ-SKETCH, [PRECISE STATEMENT](#)). *Assume the conjecture that the uniform edge distribution maximizes the ℓ -orientability threshold among all distributions (Detailed version in Conjecture 15). Then, every high-probability-successful sketch $\mathcal{A}$ in this model satisfies*

$$\liminf_{d \rightarrow \infty} \frac{\mathfrak{C}_{\mathcal{A}}(d)}{d} \geq \limsup_{d \rightarrow \infty} \frac{\mathfrak{C}_{\text{XYZ}}(d)}{d}.$$

Hence, XYZ-Sketch is asymptotically optimal in this model.

We further establish two unconditional impossibility results under this model, as follows:

THEOREM 2 (BARRIER 1: ZERO-ERROR IS IMPOSSIBLE FOR $O(1)$ UPDATE, [PRECISE STATEMENT](#)). *No sketch with $\text{poly}(d)$ cells that touches only $O(1)$ cells per update can work for every valid instance without error.*

THEOREM 3 (BARRIER 2: EXACT $\mathfrak{C} = d$ IS IMPOSSIBLE, [PRECISE STATEMENT](#)). *For every fixed $k \geq 2, \ell \geq 1$, any high-probability-successful sketch $\mathcal{A}$ in this model must satisfy $\liminf_{d \rightarrow \infty} \frac{\mathfrak{C}_{\mathcal{A}}(d)}{d} > 1$. Therefore, the idealized $\mathfrak{C} = d, O(1)$ -update point in Table 1 is (unconditionally) unattainable.*

See Section 5 for the formal model, precise theorem statements and proofs.

1.4 Key Contributions & Paper Organization

- We propose XYZ-Sketch, breaking the longstanding trade-off of the set reconciliation problem. Technically, we first introduce high-capacity cells, implemented using the D-RFR technique we developed. It also represents the first practical application of ℓ -peeling in set reconciliation, to the best of our knowledge.

³Although some sketches may have write supports larger than constant or variable sketch lengths, they are also canonical and their write locations are still fixed.

- We formulate the broad fixed-support canonical model for the set reconciliation problem. We prove that, under a natural conjecture, XYZ-Sketch is optimal in this model.
- We implement XYZ-Sketch in C++ and release the source code through [our repository](#). We also conduct experiments demonstrating its practical advantages.

The remainder of this paper is organized as follows. Section 2 introduces the relevant background. Section 3 presents high-capacity cells and D-RFR, which, together with spatial coupling, forms XYZ-Sketch. Section 4 analyzes its complexity, reliability, and parameter choices. Section 5 gives the barriers and conditional optimality results. Section 6 presents the experimental evaluation, and Section 7 concludes.

2 Background

2.1 Conventions

In real-world scenarios, the exact difference size may not be known in advance. To address the issue, prior work represented by Strata Estimator [9] estimates the difference size by exchanging only $O(\log V)$ data words in one round. We therefore focus on the setting where both parties know an upper bound on the difference size, which we denote by d . Given $A, B \subseteq \mathcal{U}^*$ satisfying $|(A \setminus B) \cup (B \setminus A)| \leq d$, the decoder must recover both $A \setminus B$ and $B \setminus A$. We additionally require $d \leq V/2$; otherwise, the problem is of little practical (and also theoretical) interest.

For simplicity, we identify the universe with a prime field⁴ $\mathcal{U} = \mathbb{F}_p$, where $p = V$. Let $\mathcal{U}^* := \mathcal{U} \setminus \{0\}$. Throughout the paper, we assume that $A, B \subseteq \mathcal{U}^*$. This does not make much difference, as Alice can always send a single bit to Bob indicating whether $0 \in A$. We assume that arithmetic on one field element costs $O(1)$ time.

We measure communication in $\mathcal{U}$ -word equivalents. Formally, a sketch has communication cost $\mathfrak{C}(d)$ if, as $V \rightarrow \infty$, it occupies $(\mathfrak{C}(d) + o(1)) \log_2 V$ bits. This accounting includes all stored auxiliary information, such as counters and fingerprints, but excludes shared public randomness (e.g. hashing functions). The precise bit-level accounting, including counters, fingerprints, and other metadata, is given in Section 4.

We denote $[n] := \{1, 2, \dots, n\}$ for $n \geq 0$ with the special case $[0] = \emptyset$. For a finite set A and an integer $d \geq 0$, we write $\binom{A}{d} := \{B \subseteq A : |B| = d\}$ and $\binom{A}{\leq d} := \{B \subseteq A : |B| \leq d\}$. For a map $f : \mathcal{U}^* \rightarrow \{-1, 0, 1\}$, we define $\text{supp}(f) := \{v : f(v) \neq 0\}$. The expression *with high probability* (w.h.p. for short) is used when the probability is $1 - o(1)$.

2.2 Hypergraphs and Thresholds

A *hypergraph* is defined as a pair $H = (\mathcal{V}, \mathcal{E})$, where every *hyperedge* $e \in \mathcal{E}$ is a subset of $\mathcal{V}$. We allow repeated hyperedges. For a vertex $v \in \mathcal{V}$, let $\deg_H(v)$ denote the number of hyperedges incident to v , counted with multiplicity.

An ℓ -*peeling step* selects a vertex v satisfying $1 \leq \deg_H(v) \leq \ell$, and removes all hyperedges incident to v . A hypergraph is called ℓ -*peelable* if repeated ℓ -peeling removes every hyperedge. When $\ell = 1$, this is the usual singleton-peeling process.

An ℓ -*orientation* of H is a map $\phi : \mathcal{E} \rightarrow \mathcal{V}$ such that

$$\forall v \in \mathcal{V}, |\phi^{-1}(v)| \leq \ell \quad \text{and} \quad \forall e \in \mathcal{E}, \phi(e) \in e$$

If such a map exists, then H is ℓ -*orientable*. Every ℓ -peelable hypergraph is ℓ -orientable, as we can orient each hyperedge to the vertex that removes it in a peeling sequence. Note that the converse

⁴Our construction can also be implemented over an extension field such as $\mathbb{F}_{2^r}$, but this requires more involved field arithmetic. We thus use $\mathbb{F}_p$ in both the presentation and the implementation.

does not hold in general. Moreover, by Hall's theorem, H is ℓ -orientable iff every submultiset $F \subseteq E$ satisfies $|F| \leq \ell \left| \bigcup_{e \in F} e \right|$.

Define the k -uniform random-hypergraph as follows: let $H_{n,m}^{(k)}$ denote the hypergraph on n vertices formed by sampling m hyperedges independently and uniformly from $\binom{[n]}{k}$. Its edge density is $c := \frac{m}{n}$. For a family of random hypergraphs $(G_{n,c})_{n \in \mathbb{N}, c \geq 0}$, we say that c^* is a threshold for a property $\mathcal{P}$ if

$$c^* := \sup \left\{ c \geq 0 : \lim_{n \rightarrow \infty} \Pr[G_{n,c} \in \mathcal{P}] = 1 \right\}.$$

For convenience, let $c_{k,\ell}^{\text{peel}}$ and $c_{k,\ell}^{\text{orient}}$ denote the thresholds for ℓ -peelability and ℓ -orientability, respectively, of $H_{n,\lfloor cn \rfloor}^{(k)}$. Intuitively, when $c < c_{k,\ell}^{(\star)}$ where $\star \in \{\text{peel}, \text{orient}\}$, the relevant property holds w.h.p., whereas above it the property fails w.h.p.. [26]

2.3 Polynomial Arithmetic

To avoid confusion between X and χ , we follow the convention of using Z as the formal variable for polynomials. The following definitions are used:

- For a nonzero polynomial $G \in \mathbb{F}_p[Z]$, $F \bmod G$ denotes the unique remainder of degree smaller than $\deg G$ obtained by dividing F by G .
- We write $F \equiv G \pmod{H}$ if H divides $F - G$.
- $\gcd(F, G)$ denotes the monic greatest common divisor of F and G .
- A nonzero polynomial is *monic* iff its leading coefficient is 1.

For $F, G \in \mathbb{F}_p[Z]$ with $G(0) \neq 0$, we write $\frac{F(Z)}{G(Z)} \bmod Z^\ell$ for the unique polynomial $R(Z)$ with $\deg R < \ell$ satisfying $R(Z)G(Z) \equiv F(Z) \pmod{Z^\ell}$. Such an R exists uniquely because $G(0) \neq 0$ implies that G is invertible modulo Z^ℓ . In particular, for every $x \in \mathcal{U}^*$, the polynomial $Z - x$ is invertible modulo Z^ℓ . Hence products with negative exponents, such as

$$\prod_x (Z - x)^{f(x)} \bmod Z^\ell, \quad f(x) \in \{-1, 0, 1\},$$

are well-defined.

For any $S \subseteq \mathcal{U}^*$, define its *characteristic polynomial* as $\chi_S(Z) := \prod_{x \in S} (Z - x)$. It is clear that:

- χ_S is monic, and $\deg \chi_S = |S|$.
- The root set of χ_S is exactly S .
- if $S \cap T = \emptyset$, then χ_S and χ_T are coprime, and $\chi_{S \cup T} = \chi_S \chi_T$.

2.4 Related Work

We briefly review existing set reconciliation work here, also demonstrating that most prior work share the abstraction of *fixed-support* and *canonical* in our barrier model.

2.4.1 Algebraic Methods. Algebraic reconciliation methods are rooted in Error Correcting Codes (ECC). Characteristic-polynomial schemes represented by CPI[18], are essentially variants of the Reed-Solomon Code[23], whereas PinSketch[8] and MiniSketch[4] are variants of BCH codes[5, 13]. Due to their algebraic properties, these sketches can be viewed as *global*: each update modifies almost all entries. Moreover, CPI may require field extensions, whereas PinSketch and Minisketch recovers only the symmetric difference and cannot distinguish $A \setminus B$ and $B \setminus A$.⁵

⁵Recovering the direction of each item typically requires maintaining an additional membership-query data structure. It may be easy to do so when the full set is static and already stored, but less suitable for an one-pass streaming sketch, e.g. in network measurement and packet loss detection scenarios.

Our *Degree-aware Rational Function Reconstruction* (D-RFR for short) follows the same coding-theoretic tradition, but takes a different representation: we directly maintain the coefficients of a characteristic polynomial truncated modulo Z^ℓ over the native field, instead of evaluations on some points. Thus, D-RFR avoids the auxiliary-field overhead of CPI while retaining directional information absent from the PinSketch/MiniSketch which use fields characteristic 2.

2.4.2 Hashing-based Methods. Hashing-based methods, represented by IBLT [12], Difference Digest [9], and Graphene [22], map each item to a constant number of cells and maintain information such as signed counts, key sums, and checksums. After subtraction, the decoder repeatedly identifies a cell containing exactly one residual item, recovers that item, and removes it from related cells.

This peeling-based architecture supports $O(1)$ local updates, but its success is limited by the singleton-peelability threshold of the induced random hypergraph, resulting in a higher space overhead. Our XYZ-Sketch retains this outer architecture while replacing singleton recovery with exact recovery of up to ℓ signed residual items within one cell. The inner structure of a high-capacity cell is implemented via D-RFR.

2.4.3 Hybrid, Rateless, and Zero-Error Schemes. In fact, there have been attempts to combine the two routes. Parity Bitmap Sketch [11] combines ECC with hashing. It targets a different trade-off from approaching the local-sketch limit and is more space-efficient than IBLT practically (but not theoretically). A recent hybrid construction [3] augments a standard singleton-peeling IBLT with a global BCH stash that corrects the residual after peeling, yielding substantially stronger reliability. In contrast, XYZ-Sketch changes the peeling rule itself: by introducing ℓ -peeling, we break the singleton-peeling threshold for hypergraphs.

Rateless IBLT and rate-compatible schemes transmit an extendable sequence of coded cells, avoiding the need to know the difference size d in advance [28]; they address protocol adaptivity rather than the communication limit of one fixed persistent sketch. For the strict zero-error requirement, IBLTs with listing guarantees [20] construct d -decodable incidence matrices that recover every set of at most d items, while CertainSync [14] reveals rateless prefixes of such constructions to guarantee reconciliation with certainty. Note that these zero-error guarantees do not contradict our barrier 1: their communication cost strongly depends on the universe size V , typically $\Omega_k(V^{1/k} \log V)$ bits. Finally, robust set reconciliation [6] changes the problem itself by allowing sufficiently nearby elements to match, rather than requiring recovery of the exact symmetric difference.

2.4.4 Threshold Saturation and Spatial Coupling. For independently hashed hypergraphs, the ℓ -peelability threshold lies strictly below the ℓ -orientability threshold [10], which marks the feasibility limit for assigning each edge to an incident cell of capacity ℓ . Spatial coupling was developed in coding theory by [16] to achieve *threshold saturation* for iterative LDPC decoding. Walzer transferred this mechanism to hashing-based data structures in [26] and proved that, as the coupling length grows, the ℓ -peelability threshold of a spatially coupled hypergraph approaches the ℓ -orientability threshold of the corresponding fully random instance. We use the threshold-saturation results as a theoretical input for our proof.

3 Design of XYZ-Sketch

In this section, we present XYZ-Sketch. For details, refer to the pseudocode in the Appendix E or the C++ implementation in our repository <https://github.com/djwj233/XYZ-Sketch>.

3.1 Framework: High-Capacity Cells and ℓ -Peeling

3.1.1 Structure of the XYZ-Sketch. The sketch $\mathcal{B}$ contains M cells, numbered from 1 to M . The i -th cell is denoted by $\mathcal{B}[i]$. We fix two **constant** parameters: 1) k , the number of cells touched

by each element; 2) ℓ , the capacity of each cell. We use k mutually independent hash functions $h_1, \dots, h_k : \mathcal{U}^* \rightarrow [M]$.⁶

The detailed implementation of each cell is sophisticated and deferred to Section 3.2; here, we describe its functionality solely through its interface. A cell state C maintains a map $f_C : \mathcal{U}^* \rightarrow \{-1, 0, 1\}$. We denote a cell *pure* iff it contains at most ℓ nonzero entries, i.e. $|\text{supp}(f_C)| \leq \ell$. It supports the following operations:

- **Update**(C, x, σ), where $\sigma \in \{+1, -1\}$, which means $f_C(x) \leftarrow f_C(x) + \sigma$;
- **Subtract**(C, C'), which returns the state representing $f_C - f_{C'}$, thereby canceling elements shared by the two cells and retaining their differences;
- **Decode _{ℓ}** (C), which returns exactly $\{(x, f_C(x)) : x \in \mathcal{U}^*, f_C(x) \neq 0\}$ when C is pure, and otherwise it may return $\perp$ or an arbitrary output.

Note that all operations require the resulting value to remain in $\{-1, 0, 1\}$; otherwise, the behavior is undefined.

3.1.2 The Encoding & Decoding Algorithm. During the encoding phase, when an element $x \in \mathcal{U}^*$ is inserted, we first compute its k hash locations $h_1(x), \dots, h_k(x)$ and remove duplicates.⁷ For every remaining location j , the encoder performs $\mathcal{B}[j] \leftarrow \text{Update}(\mathcal{B}[j], x, +1)$.

During the decoding phase, the decoder first subtracts the two sketches cell by cell, obtaining the sketch $\mathcal{D}$ that

$$\mathcal{D}[i] := \text{Subtract}(\mathcal{B}_A[i], \mathcal{B}_B[i]).$$

This cancels all common elements and leaves only the differences: an element in $A \setminus B$ has sign $+1$, while an element in $B \setminus A$ has sign -1 .

To detect (non-empty) pure cells, the decoder tentatively runs **Decode _{ℓ}** on every cell. The returned candidate must then pass verification. In particular, we use *rehashing verification*: every recovered element x must indeed be mapped to the cell being verified, i.e., the cell must belong to $\Gamma(x) := \{h_1(x), \dots, h_k(x)\}$. A verified cell is treated as pure.

The decoder repeatedly finds a verified pure cell, recovers the signed differences in it, and eliminates their effects from all incident cells. More precisely, for every recovered (x, σ) and every $j \in \Gamma(x)$, it performs $\mathcal{D}[j] \leftarrow \text{Update}(\mathcal{D}[j], x, -\sigma)$. This may create new pure cells, and the process continues until $\mathcal{D}$ is empty or no verified pure cell remains. The former case indicates success, while the latter indicates failure. Detailed pseudocode is provided in the Appendix E.

Connection to the ℓ -Peeling Process. Consider the hypergraph whose vertices are the cells and whose hyperedges are the difference elements, with x incident to the cells in $\Gamma(x)$. It can be seen that, the above decoding procedure is exactly the ℓ -peeling process on this hypergraph. Therefore, assuming that verification accepts exactly the pure cells, decoding succeeds iff the induced hypergraph is ℓ -peelable.

3.2 Implementing the High-Capacity Cell via D-RFR

In this part, we propose the *Degree-aware Rational Function Reconstruction* technique (D-RFR), and implement the high-capacity cell interface in Section 3.1 via it.

3.2.1 D-RFR. Rational Function Reconstruction (RFR) is a classical Padé-approximation problem. D-RFR adapts it to our truncated representation by exploiting the known degree difference and the monicity of characteristic polynomials. We obtain the following guarantee.

⁶In our implementation, we use MurmurHash[2], which is a widely adopted hashing scheme.

⁷The version without de-duplication indeed has the same asymptotic peeling threshold. It avoids the de-duplication step and can be slightly faster in practice.

THEOREM 4 (THE D-RFR TECHNIQUE). *Let $R(Z)$ be a polynomial with $\deg R < \ell$, and let $m \in [-\ell, \ell]$. Suppose that there exist monic coprime polynomials $F(Z)$ and $G(Z)$ satisfying*

$$G(0) \neq 0, \quad \deg F + \deg G \leq \ell, \quad \deg F - \deg G = m,$$

$$\frac{F(Z)}{G(Z)} \bmod Z^\ell = R(Z).$$

Then F and G are uniquely determined by (R, m) . Moreover, D-RFR recovers them from (R, m) in $O(\ell \log^2 \ell)$ field operations.

Method overview. Our D-RFR saves one coefficient: modulo Z^ℓ suffices, whereas a direct application of standard RFR would require modulo $Z^{\ell+1}$ [7, 15]. Technically, given (R, m) , D-RFR uses elementary algebraic transformations, together with the known degree difference and monicity, to reduce the problem to a standard balanced RFR instance, which is solved by half-GCD. Since the algorithm details are complicated yet somewhat tedious, we defer the reconstruction procedure to the Appendix C and prove the uniqueness part only.

PROOF. Suppose that (F_0, G_0) and (F_1, G_1) are two valid solutions. We have

$$F_0(Z)G_1(Z) \equiv F_1(Z)G_0(Z) \pmod{Z^\ell}.$$

Write $a_j := \deg F_j$ and $b_j := \deg G_j$ for $j \in \{0, 1\}$. Since $a_j - b_j = m$ and $a_j + b_j \leq \ell$, we have

$$\deg(F_0G_1) = a_0 + b_1 = m + b_0 + b_1 = a_1 + b_0 = \deg(F_1G_0) \leq \ell.$$

Both products are monic, so their leading terms cancel. Hence $\deg(F_0G_1 - F_1G_0) < \ell$. Since this difference is divisible by Z^ℓ , it must be zero. Thus, $F_0G_1 = F_1G_0$. As F_0 and G_0 are coprime, this implies $F_0 \mid F_1$ and $G_0 \mid G_1$. Since F_1 and G_1 are also coprime, the common quotient must be a constant; monicity then gives $F_0 = F_1$ and $G_0 = G_1$. $\square$

3.2.2 Cell State. Recall that each cell state C represents a map $f_C : \mathcal{U}^* \rightarrow \{-1, 0, 1\}$. It stores two fields: count field and polynomial field. Specifically,

$C = (c_C, P_C)$, where

$$c_C := \sum_{x \in \mathcal{U}^*} f_C(x) \pmod{2\ell + 1}, \quad P_C(Z) := \prod_{x \in \mathcal{U}^*} (Z - x)^{f_C(x)} \bmod Z^\ell.$$

Since $0 \notin \mathcal{U}^*$, every factor $Z - x$ is invertible modulo Z^ℓ . Thus, for every legal call of $\text{Update}(C, x, \sigma)$, the cell updates

$$c_C \leftarrow c_C + \sigma \pmod{2\ell + 1}, \quad P_C(Z) \leftarrow P_C(Z)(Z - x)^\sigma \bmod Z^\ell.$$

Likewise, $\text{Subtract}(C, C')$ returns the state representing $f_C - f_{C'}$: its count field is $c_C - c_{C'}$, and its polynomial field is $P_C(Z)/P_{C'}(Z) \bmod Z^\ell$.

3.2.3 Decoding. The decoding process $\text{Decode}_\ell(C)$ is a bit more sophisticated. Given a cell state C , let $m \in [-\ell, \ell]$ be the unique integer satisfying $m \equiv c_C \pmod{2\ell + 1}$. We run D-RFR on $(P_C(Z), m)$. If it does not return a valid pair of monic coprime polynomials (F, G) satisfying Theorem 4 or F, G does not split into linear factors, the decoder returns $\perp$. Otherwise, it factors F and G and returns the roots of F with sign +1 and the roots of G with sign -1.

Claim. If C is pure, then $\text{Decode}_\ell(C)$ returns exactly the remaining signed differences in C .

PROOF. In particular, if C is pure, let $\Delta^+ := \{x : f_C(x) = +1\}$, $\Delta^- := \{x : f_C(x) = -1\}$. By definition we have $|\Delta^+| + |\Delta^-| \leq \ell$ and

$$P_C(Z) = \frac{\chi_{\Delta^+}(Z)}{\chi_{\Delta^-}(Z)} \bmod Z^\ell, \quad m = |\Delta^+| - |\Delta^-|.$$

Therefore, D-RFR returns $F = \chi_{\Delta^+}$ and $G = \chi_{\Delta^-}$, so the decoder recovers exactly the remaining signed differences. $\square$

3.2.4 Fingerprint Verification. Rehashing verification rejects a decoded candidate whenever one of its elements is not mapped to the cell being tested. To further reduce the false-positive rate (i.e. probability that a non-pure cell is verified as pure), we augment each cell with an additional *fingerprint* field.

Let $h_{\text{fp}} : \mathcal{U}^* \rightarrow \mathbb{Z}_q$ be a random hashing function independent of h_i , where q is odd. For a cell state C , we store

$$\text{fp}_C := \sum_{x \in \mathcal{U}^*} f_C(x) h_{\text{fp}}(x).$$

This field is maintained together with the other cell fields: each call to $\text{Update}(C, x, \sigma)$ adds $\sigma h_{\text{fp}}(x)$, and $\text{Subtract}(C, C')$ subtracts the two fingerprint values.

Suppose that $\text{Decode}_\ell(C)$ returns a candidate signed map $\widehat{f}_C$. After it passes rehashing verification, we additionally check whether

$$\sum_{x \in \mathcal{U}^*} \widehat{f}_C(x) h_{\text{fp}}(x) = \text{fp}_C.$$

If the check fails, simply return $\perp$. A pure cell always passes this check. Intuitively, if $\widehat{f}_C \neq f_C$, the check holds with probability at most $1/q$, which will be proved later. Thus, the fingerprint field reduces the probability that an incorrect candidate is used during peeling.

3.3 Spatial Coupling

Spatial coupling originated in coding theory as a technique for improving iterative-decoding thresholds, and was later adapted to hashing-based data structures in [26]. Rather than mapping the k locations of an element independently over the whole sketch, spatial coupling places them in a common short range. Fix a coupling parameter $z > 0$. Let $g_0 : \mathcal{U}^* \rightarrow [0, z]$ and $g_1, \dots, g_k : \mathcal{U}^* \rightarrow [0, 1]$ be mutually independent uniform random functions. For every $x \in \mathcal{U}^*$, define

$$h'_i(x) := 1 + \left\lfloor \frac{g_0(x) + g_i(x)}{z + 1} M \right\rfloor, \quad i \in [k].$$

Conditioned on $g_0(x)$, all k locations of x lie in an interval of length approximately $M/(z + 1)$. Cells near the two ends have smaller expected loads, and hence provide the initial pure cells from which ℓ -peeling propagates through the sketch.

Circular Trick. For practical performance, we propose a circularized placement rule, under which a coupled range may wrap around from the last cell to the first. Specifically, we let $g_0 : \mathcal{U}^* \rightarrow [0, z + a]$, $a \in [0, 1]$ and $g_1, \dots, g_k : \mathcal{U}^* \rightarrow [0, 1]$ are mutually independent uniform random functions, and replace the above mapping by

$$h_i^{(a)}(x) := 1 + \left\lfloor \left(\frac{g_0(x) + g_i(x)}{z + 1} \bmod 1 \right) M \right\rfloor, \quad i \in [k].$$

4 Theoretical Analysis

Throughout this section, k and ℓ are fixed constants. The size of the fingerprint field is denoted by $q \geq 3$. For convenience, we use $q = 1$ to denote the version without a fingerprint.

4.1 Complexity

PROPOSITION 5 (COMPUTATIONAL COMPLEXITY). *Suppose $M = \Theta(d)$ and $|(A \setminus B) \cup (B \setminus A)| \leq d$, XYZ-Sketch supports each update in $\Theta(1)$ time and performs decoding in worst-case $\Theta(d \log V)$ expected time.⁸*

PROOF. For each inserted item x , the algorithm updates at most k cells. Within one cell, it updates a signed counter c , a fingerprint fp , and the polynomial $P(Z) \leftarrow P(Z)(Z - x) \bmod Z^\ell$. The latter operation takes $\Theta(\ell)$ field operations. Hence one update costs $\Theta(k\ell) = \Theta(1)$ time.

For decoding, we first verify all M cells and insert every currently pure cell into a queue. Whenever a cell is decoded, it recovers at most ℓ residual items. Each recovered item is removed from the k cells in its support, which can be recomputed from the item itself. Since every differing item is recovered at most once, the total number of such cell updates is at most kd .

As a cell is tested only initially or after one of its incident items has been removed, the total number of decoding attempts is at most $M + kd = O(d)$. Conversely, successfully decoding an instance with d different items requires at least $d/\ell = \Omega(d)$ nonempty cell decodings. For a fixed ℓ , one decoding attempt invokes D-RFR and factors constant-degree polynomials over $\mathbb{F}_p$, which takes $\Theta(\log V)$ expected time. Overall, the total expected decoding time is $\Theta(d \log V)$. $\square$

PROPOSITION 6 (BIT-LEVEL SPACE COMPLEXITY). *One XYZ-Sketch uses*

$$M (\ell \lceil \log_2 V \rceil + \lceil \log_2 (2\ell + 1) \rceil + \lceil \log_2 q \rceil)$$

bits, excluding the shared descriptions of the hash functions.

Equivalently, its size is $M(\ell \log V + \log q + O(\log \ell))$ bits.

PROOF. Each cell stores a polynomial field $P_C(Z) \bmod Z^\ell$, represented by its ℓ coefficients in $\mathcal{U}$, which requires $\ell \lceil \log_2 p \rceil$ bits. Its count field lies in $\mathbb{Z}_{2\ell+1}$ and requires $\lceil \log_2 (2\ell + 1) \rceil$ bits. Its fingerprint field lies in $\mathbb{Z}_q$ and requires $\lceil \log_2 q \rceil$ bits. Multiplying the resulting per-cell cost by the number M of cells proves the claim. $\square$

4.2 Pure-Cell Verification

In this part, we show that in the random label setting discussed below, rehashing verification alone yields a global false-positive probability of $O(M^{1-\ell})$. To further suppress this error at small d , it is suggested to use an independent fingerprint additionally, reducing the bound to $O(M^{1-\ell}/q)$. Throughout this subsection, we assume $M = \Theta(d)$.

Random Label. Fix $d_+, d_- \geq 0$ with $d_+ + d_- \leq d$. We analyze the model in which $(A \setminus B, B \setminus A)$ is sampled uniformly from all disjoint ordered pairs of sizes (d_+, d_-) . Note that arbitrary input instances can be reduced to the random setting simply by applying a random public bijection $\phi : \mathcal{U}^* \rightarrow \mathcal{U}^*$ before sketching and applying ϕ^{-1} after decoding. We treat ϕ , like the hashing functions h , as shared public randomness and exclude its description from communication.

⁸As in CPI and PinSketch, the expectation arises solely from the randomized root-finding routine; all remaining operations in the decoding procedure are deterministic.

Analysis of Rehashing. Let $N := |\mathcal{U}^*| = V - 1$. We equivalently expose the randomness as follows. Each original element x first receives its placement support $\Gamma(x)$, and an independent uniformly random bijection ϕ then assigns its field label. For every cell j , $\Pr[j \in \Gamma(x)] \leq C_0/M$, where the constant C_0 depends only on the fixed placement method. Assume throughout that $M = \Theta(d)$; $d \leq N/2$. For a signed map $f : \mathcal{U}^* \rightarrow \{-1, 0, 1\}$, define $\Psi_f(Z) := \prod_{x \in \mathcal{U}^*} (Z - \phi(x))^{f(x)} \bmod Z^\ell$.

LEMMA 7. *Let $f \neq g$ be fixed signed maps with $|\text{supp}(f) \cup \text{supp}(g)| \leq d + \ell$. Then*

$$\Pr_{\phi}[\Psi_f(Z) = \Psi_g(Z)] = O(N^{-\ell}).$$

We defer the proof to Appendix B due to its complexity.

LEMMA 8. *The probability that a non-pure cell passes rehashing verification at any point before the first erroneous peeling step is $O(M^{1-\ell})$.*

PROOF. For each cell j , let $L_j := |\Gamma^{-1}(j) \cap ((A \setminus B) \cup (B \setminus A))|$, where $\Gamma^{-1}(j) := \{x : j \in \Gamma(x)\}$. Before the first erroneous step, the residual content of cell j is obtained by removing a subset of these L_j elements and has at most 2^{L_j} possible values.

If a candidate (S, T) passes rehashing in cell j , then $S \cup T \subseteq \Gamma^{-1}(j)$ and $|S| + |T| \leq \ell$. Hence, the number of such (S, T) is at most $\sum_{t=0}^{\ell} 2^t \binom{|\Gamma^{-1}(j)|}{t} = O((1 + |\Gamma^{-1}(j)|)^\ell)$ such candidates. Therefore, Lemma 7 and a union bound give

$$\Pr[\text{an incorrect } (S, T) \text{ passes rehashing at any point}] \leq O(N^{-\ell}) \sum_{j=1}^M 2^{L_j} (1 + |\Gamma^{-1}(j)|)^\ell.$$

The variables L_j and $K_j := |\Gamma^{-1}(j) \setminus ((A \setminus B) \cup (B \setminus A))|$ are independent Poisson-binomial variables with means $O(d/M) = O(1)$ and $O(N/M)$, respectively. Since ℓ is fixed, standard moment bounds imply

$$\begin{aligned} \mathbb{E}[2^{L_j} (1 + L_j)^\ell] &\leq O(\mathbb{E}[3^{L_j}]) = O(1) \\ \mathbb{E}[2^{L_j} (1 + L_j + K_j)^\ell] &\leq \mathbb{E}[2^{L_j} (1 + L_j)^\ell] \cdot \mathbb{E}[(1 + K_j)^\ell] = O((1 + N/M)^\ell) \end{aligned}$$

Consequently, the probability is at most $O\left(N^{-\ell} M \left(1 + \frac{N}{M}\right)^\ell\right) = O(M^{1-\ell})$. $\square$

LEMMA 9. *For every fixed pair of signed maps $f \neq g$, their fingerprints agree with probability q^{-1} , independently of the polynomial collision event.*

PROOF. The case of $q = 1$ is trivial. Otherwise, choose x_0 with $f(x_0) \neq g(x_0)$. Fingerprint equality is equivalent to

$$\sum_x (f(x) - g(x)) h_{\text{fp}}(x) = 0.$$

The coefficient of $h_{\text{fp}}(x_0)$ belongs to $\{-2, -1, 1, 2\}$ and is a unit in $\mathbb{Z}_q$ because q is odd. After fixing all other fingerprint values, there is exactly one value of $h_{\text{fp}}(x_0)$ which satisfies the equation. $\square$

PROPOSITION 10. *The probability that a false positive occurs during decoding is $O(M^{1-\ell}/q)$.*

PROOF. In the union bound of Lemma 8, every fixed incorrect pair incurs an additional independent factor q^{-1} by Lemma 9. The same counting argument therefore gives $O(M^{1-\ell}/q)$. $\square$

4.3 Space Overhead Analysis

The intuition of spatial coupling is as follows: without coupling, peeling succeeds only below the threshold $c_{k,\ell}^{\text{peel}}$; spatial coupling raises this threshold arbitrarily close to the larger orientability threshold $c_{k,\ell}^{\text{orient}}$. We analyze only the conventional coupling rule h'_i in Section 3.3 here; the circular trick will be discussed later.

LEMMA 11 (COUPLED PEELING). *Fix k, ℓ with $k \geq 2$ and $k + \ell \geq 4$, and let $c < c_{k,\ell}^{\text{orient}}$. For every sufficiently large fixed $z > 0$, let*

$$M = \left\lceil \left(1 + \frac{1}{z}\right) \frac{d}{c} \right\rceil.$$

Then, for every $t \leq d$, the hypergraph induced by the spatial coupling rule of t independently sampled edges is ℓ -peelable w.h.p..

PROOF. [26] proves that such coupled hypergraph on M vertices with $\lfloor cMz/(z+1) \rfloor$ independently sampled hyperedges is ℓ -peelable w.h.p.. Let $m' := \lfloor \frac{cMz}{z+1} \rfloor$. Our choice of M ensures that $m' \geq d \geq t$. Generate an auxiliary coupled hypergraph with m' independent hyperedges and retain its first t edges. These t edges have exactly the distribution induced by our placement rule. The claim follows directly from that ℓ -peelability is preserved under deleting hyperedges. $\square$

THEOREM 12 (XYZ-SKETCH APPROACHES THE ORIENTABILITY THRESHOLD). *Assume $d \leq V/2$ and the random-label setting of Section 4.2. Fix constants $k, \ell \geq 2$. There exist sequences $M_d, z_d \in \mathbb{N}$, such that, when XYZ-Sketch is instantiated with them, it solves every instance with $|(A \setminus B) \cup (B \setminus A)| \leq d$ w.h.p. and satisfies*

$$\limsup_{d \rightarrow \infty} \frac{\mathfrak{C}(d)}{d} \leq \frac{\ell}{c_{k,\ell}^{\text{orient}}}.$$

The parameter schedules M_d and z_d depend only on d, k, ℓ , and not on the input instance.

PROOF. For each integer $j \geq 1$, choose $\eta_j > 0$ such that $(1 + \eta_j)^2 \frac{\ell}{c_{k,\ell}^{\text{orient}}} \leq \frac{\ell}{c_{k,\ell}^{\text{orient}}} + \frac{1}{j}$. Choose $c_j < c_{k,\ell}^{\text{orient}}$ that $\frac{\ell}{c_j} < (1 + \eta_j) \frac{\ell}{c_{k,\ell}^{\text{orient}}}$. By the result for spatial coupling [26], there exists a fixed coupling parameter z_j such that $1 + \frac{1}{z_j} < 1 + \eta_j$ and the coupled hypergraph with density c_j is ℓ -peelable w.h.p..

For fixed j , let $M_{j,d} := \left\lceil \left(1 + \frac{1}{z_j}\right) \frac{d}{c_j} \right\rceil$. By Lemma 11, the probability that the induced hypergraph is not ℓ -peelable is $o(1)$ as $d \rightarrow \infty$. Moreover, when $q = 1$, meaning that even discarding fingerprints, Proposition 10 gives a false-positive probability of $O\left(M_{j,d}^{1-\ell}/q\right) = o(1)$, since $\ell \geq 2$ and $M_{j,d} = \Theta(d)$.

Hence, for every j , there exists D_j such that, for all $d \geq D_j$, the decoding failure probability under the parameters (M_j, z_j) is at most $1/j$. Choose the sequence $D_1 < D_2 < \dots$ strictly increasing and let $j(d) := \max\{j : D_j \leq d\}$, then instantiate XYZ-Sketch with $c_d := c_{j(d)}$, $z_d := z_{j(d)}$, $M_d := M_{j(d),d}$, $q = 1$. Since $j(d) \rightarrow \infty$, the decoding failure probability is at most $1/j(d) = o(1)$. Finally,

$$\frac{\mathfrak{C}(d)}{d} = \frac{M_d \ell}{d} < (1 + \eta_{j(d)})^2 \frac{\ell}{c_{k,\ell}^{\text{orient}}} + \frac{\ell}{d} \implies \limsup_{d \rightarrow \infty} \frac{\mathfrak{C}(d)}{d} \leq \frac{\ell}{c_{k,\ell}^{\text{orient}}}.$$

$\square$

THEOREM 13 (MAIN GUARANTEE). *Assume $d \leq V/2$ and use the public random relabeling described in Section 4.2. For all constants $\varepsilon, \delta > 0$, there exist fixed parameters k, ℓ such that, for all sufficiently large d , XYZ-Sketch solves every instance with $|(A \setminus B) \cup (B \setminus A)| \leq d$ with probability at least $1 - \delta$ and satisfies:*

- $O(1)$ per-item update time and expected $O(d \log V)$ decoding time;
- At most $\lceil (1 + \epsilon)d \log_2 V \rceil$ bits of communication, or $\mathfrak{C} \leq (1 + \epsilon)d$.

The constants hidden in the $O(\cdot)$ notation may depend on ϵ and δ , but not on d or V .

PROOF. As $\ell/c_{k,\ell}^{\text{orient}} \rightarrow 1$ when k and ℓ increase[10], we may fix constant $k, \ell \geq 2$ such that $\ell/c_{k,\ell}^{\text{orient}} < 1 + \frac{\epsilon}{4}$. Apply Theorem 12 with these k, ℓ , which yields parameter schedules M_d, z_d such that XYZ-Sketch succeeds w.h.p. and

$$\limsup_{d \rightarrow \infty} \frac{\mathfrak{C}(d)}{d} \leq \frac{\ell}{c_{k,\ell}^{\text{orient}}} < 1 + \frac{\epsilon}{4}.$$

Hence, for all sufficiently large d , $\mathfrak{C}(d) \leq (1 + \frac{\epsilon}{2})d$. Moreover, the failure probability is $o(1)$, and is therefore at most δ for all sufficiently large d . The complexity part is already shown in Proposition 5. Finally, Proposition 6 gives

$$M_d \ell \log_2 V + O(M_d) \leq \left(1 + \frac{\epsilon}{2}\right) d \log_2 V + O(d) \leq (1 + \epsilon)d \log_2 V.$$

The last inequality holds for sufficiently large d since $V \geq 2d$. $\square$

4.4 Heuristic Choices for Parameter a and z

This subsection gives an informal explanation for our choices of a and z . The discussion is heuristic rather than rigorous, and aims to derive reasonable parameter choices that can be calibrated experimentally. A sharp finite-length analysis remains challenging even for extensively studied spatially coupled LDPC codes in coding theory and is beyond the scope of this work.[21, 24]

Technically, we combine Walzer's density-evolution analysis[26] with the OU's finite-length scaling heuristic [21] from coding theory, and adapt them to our circular placement rule and capacity- ℓ cells.

4.4.1 Choosing a . We first explain the choice of a in our circular trick. Let $L := z + 1$ and $\mathbb{T}_L := \mathbb{R}/L\mathbb{Z}$ be the circle of circumference L , with all intervals below understood modulo L . For each edge, its *anchor* s is sampled uniformly from $[0, z + a)$, and its k *endpoints* are sampled independently from the unit interval $[s, s + 1] \subseteq \mathbb{T}_L$. Following Walzer [26], we describe the *density* at each position $u \in \mathbb{T}_L$ by $w_a(u) := |[u - 1, u] \cap [0, z + a)|$, where $|\cdot|$ denotes arc length on $\mathbb{T}_L$. Thus, $w_a(u)$ is the measure of anchor locations whose placement interval contains u . In particular, $\min_u w_a(u) = a$, attained on $u \in [0, a]$.

Let $N := M/L$. As $N \rightarrow \infty$, the number of surviving incident edges at each position is approximated by a Poisson random variable. For a measurable function $q : \mathbb{T}_L \rightarrow [0, 1]$ where $q(u)$ denotes the survival probability at position u in the current peeling round, define

$$(\mathbf{P}_{a,z,cq})(u) := Q\left(ck \int_{[u-1,u] \cap [0,z+a)} \left(\int_0^1 q(s+t) dt\right)^{k-1} ds, \ell\right),$$

where additions are modulo L and $Q(\lambda, \ell) := \Pr[\text{Pois}(\lambda) \geq \ell]$. Since this operator shares a similar form with the original one, peeling is expected to start on $[0, a]$ where w_a is minimized and then propagate toward positions of larger density.

Suppose that the operating density c_0 is close to $c_{k,\ell}^{\text{orient}}$. Since the minimum local density is ac_0 , requiring it to lie below the uncoupled peeling threshold suggests $ac_0 \lesssim c_{k,\ell}^{\text{peel}}$. We therefore choose

$$a_{k,\ell} = C \frac{c_{k,\ell}^{\text{peel}}}{c_{k,\ell}^{\text{orient}}}, \quad 0 < C < 1,$$

where C leaves a margin for finite-size fluctuations and is calibrated in Section 6.

4.4.2 Choosing z . We next explain the choice of z . We write $d = cM \frac{z+a}{z+1}$, $L = z + 1$, $N = M/L$. Following Walzer, we split the density loss to two parts. We write

$$\frac{d}{M} = c \frac{z+a}{z+1} = c \left(1 - \frac{1-a}{z+1} \right).$$

Hence, relative to the ideal density $c \rightarrow c_{k,\ell}^{\text{orient}}$, the first loss term is $\Delta_{\text{len}} = \frac{1-a}{z+1} \approx \frac{1-a}{z}$.

It remains to estimate the loss caused by finite-size fluctuations. Let s be the number of edge-removal steps, $\tau := s/N$ the normalized peeling time, and

$$A(\tau) := \left| \{v : 1 \leq \deg^{(\lfloor N\tau \rfloor)}(v) \leq \ell\} \right|$$

the number of currently pure cells. Following the finite-length scaling picture for spatially coupled codes[21, 24], let Δ denote the gap from the peeling threshold, measured in the same units as c . We assume $\mathbb{E}[A(\tau)] = \Theta(N\Delta)$ and that $A(\tau)$ has sub-Gaussian lower-tail distributions:

$$\Pr[A(\tau) - \mathbb{E}[A(\tau)] \leq -x] \leq \exp\left(-\Theta\left(\frac{x^2}{N}\right)\right).$$

As the failure event is $A(\tau) = 0$ and the peeling persists for $\Theta(z)$ units of normalized time, a union bound gives $\Pr[\text{failure}] \lesssim z \exp(-\Theta(N\Delta^2))$. Requiring this probability to be at most δ suggests $\Delta_{\text{fluc}} = \Theta\left(\sqrt{\frac{\log(z/\delta)}{N}}\right) = \Theta\left(\sqrt{\frac{z \log(z/\delta)}{M}}\right)$. Therefore, we estimate the total loss by $\kappa_1 \frac{1-a}{z} + \kappa_2 \sqrt{\frac{z \log(z/\delta)}{M}}$, where $\kappa_1, \kappa_2 > 0$ are constants. Balancing the two terms gives $z^3 \log(z/\delta) = \Theta((1-a)^2 M)$. Ignoring the slowly varying $\log z$ factor, we use

$$z_{k,\ell} = D(1 - a_{k,\ell})^{2/3} \left(\frac{M}{\log(1/\delta)} \right)^{1/3},$$

where $D > 0$ is a constant and will be calibrated experimentally.

5 Barriers

The goal of this section is to formalize a model for set reconciliation sketches and to show that, within this model, XYZ-Sketch already achieves the optimal asymptotic time-space trade-off, conditional on the extremality conjecture stated below. Additionally, we also provide unconditional impossibility results that clarify the boundaries of this model.

5.1 Our Model

We formalize a class of persistent sketches with fixed write supports. We work in the one-pass insertion-only streaming setting: elements of the input set arrive in an arbitrary order, each element is processed exactly once and cannot be revisited, and between updates the algorithm retains information only in its persistent sketch state. Fix constants $k, \ell \geq 1$. For every d, V , a *fixed-support canonical sketch family* $\mathcal{A}$ samples public randomness ω (e.g. hashing functions. After fixing it, the encoding and decoding are deterministic) and maintains M_d persistent cells, where M_d depends only on d , not on V , since the desired space overhead should be at most $O(\text{poly}(d) \log V)$. All persistent information is stored in these cells. For every fixed d , each cell has at most $V^{\ell + r_d(V)}$ possible states,

where $r_d(V) = o_V(1)$.⁹ We denote $\mathfrak{C}_{\mathcal{A}}(d) := \ell M_d$. The total persistent state therefore contains at most $(\mathfrak{C}_{\mathcal{A}}(d) + o_V(1)) \log_2 V$ bits. We impose the following natural assumptions on a practically feasible streaming algorithm:

1) Fixed-support. After fixing ω , every $x \in \mathcal{U}^*$ is assigned a predetermined write support $S_{d,V}^\omega(x) \subseteq [M_d]$. An insertion of x may inspect the entire sketch and perform arbitrary computation, but may modify only the cells in this support. The aspect that concerns us most is constant support, i.e. every $S_{d,V}^\omega(x)$ having size at most k . Unless stated otherwise, we use “fixed-support” throughout this section to refer to the constant-support setting described above.

2) Canonicity. The final sketch should depend only on the set, rather than the arriving order. Formally, the following property holds after fixing ω :

$$\forall A \subseteq \mathcal{U}^*, x \in (\mathcal{U}^* \setminus A), \quad \text{Enc}_{d,V}^\omega(A \cup \{x\}) = \text{Upd}_{d,V,x}^\omega(\text{Enc}_{d,V}^\omega(A))$$

where $\text{Upd}_{d,V,x}^\omega$ represents a single update operation of x , and $\text{Enc}_{d,V}^\omega$ represents the encoding function.

Given $\text{Enc}_{d,V}^\omega(A)$ and $\text{Enc}_{d,V}^\omega(B)$, the decoder $\text{Dec}_{d,V}^\omega$ may inspect both sketches and perform arbitrary computation to output $(A \setminus B, B \setminus A)$. Note that we impose no implementation-specific restrictions, such as requiring the decoder to employ a peeling strategy. We treat the decoder purely as a black box, imposing no constraints on its time or space complexity.

To simplify the analysis, our lower bounds use the subfamily $(A, B) = (D, \emptyset)$ where $|D| \leq d$. Intuitively, a reasonable algorithm should succeed w.h.p. on this simpler subfamily, thus it may serve as a necessary condition.¹⁰ Let $D \sim \text{Unif}(\binom{\mathcal{U}^*}{d})$. We define

$$\text{Succ}_{d,V}^\omega(\mathcal{A}) := \Pr_D \left[\text{Dec}_{d,V}^\omega(\text{Enc}_{d,V}^\omega(D), \text{Enc}_{d,V}^\omega(\emptyset)) = (D, \emptyset) \right],$$

For brevity, we may slightly abuse notation and write $\text{Dec}(\text{Enc}(D))$ for the first component of $\text{Dec}(\text{Enc}(D), \text{Enc}(\emptyset))$.

We write $\text{Succ}_{d,V}(\mathcal{A}) := \mathbb{E}_\omega[\text{Succ}_{d,V}^\omega(\mathcal{A})]$. We call a family of sketch algorithms $\mathcal{A}$ *with-high-probability successful* iff $\lim_{d \rightarrow \infty} \liminf_{V \rightarrow \infty} \text{Succ}_{d,V}(\mathcal{A}) = 1$.

5.2 Barrier 1: Zero-Error is Impossible

We first give the first barrier, which shows that zero-error recoveries with $O(1)$ time update is impossible.¹¹ The proof is a simple pigeonhole argument.

Theorem 2 (Barrier 1, Precise Version). Fix constant $\ell, k \geq 1$ and $d \geq \ell k + 1$. For all sufficiently large V , no fixed-support canonical sketch family satisfies

$$\forall A \in \binom{\mathcal{U}^*}{d}, \quad \Pr_\omega [\text{Dec}^\omega(\text{Enc}^\omega(A)) = A] = 1,$$

PROOF. Suppose for contradiction that the stated zero-error guarantee holds. Since $\binom{\mathcal{U}^*}{d}$ is finite, there exist fixed random tapes ω_0 such that $\forall A \in \binom{\mathcal{U}^*}{d}, \text{Dec}^{\omega_0}(\text{Enc}^{\omega_0}(A)) = A$. Fix ω_0 and omit them from the notation.

⁹Allowing ℓ -word cells models cache-line-sized cells, and also remains meaningful when several small-universe elements fit within one machine word. The $r_d(V)$ term accommodates auxiliary metadata, such as the fingerprints and counters used in IBLT-style sketches.

¹⁰This restriction is also natural from the canonicity: writing $P = A \setminus B$, $Q = B \setminus A$, and $C = A \cap B$, one may insert the common part C after P and Q on both sides. Conditioned on C , the resulting pair of sketches is thus a common deterministic processing of $(\text{Enc}(P), \text{Enc}(Q))$, which cannot reveal additional information about the residual difference.

¹¹Existing zero-error algorithms do not contradict our conclusion, as their persistent space generally depends on the universe size V .

There are at most $N_d := \sum_{i=0}^k \binom{M_d}{i}$ possible write supports. Hence some $R \subseteq [M_d]$ has a preimage

$$Q_R := \{x \in \mathcal{U}^* : S_{d,V}(x) = R\}$$

of size at least $(V-1)/N_d$. Let $u := |R|$ and $t := \ell u + 1$. Since $u \leq k$, we have $t \leq d$.

Fix any $F \subseteq \mathcal{U}^*$ of size $d-t$, and let $Q' := Q_R \setminus F$. For each $T \in \binom{Q'}{t}$, we can insert F first and then T by canonicity. Since every element of T modifies only the cells in R , all sketches $\{\text{Enc}(A_T) : T \in \binom{Q'}{t}\}$ agree outside R . Thus their number is at most $V^{(\ell+o_V(1))u}$.

On the other hand, $|Q'| = \Omega(V)$, and hence $\left| \binom{Q'}{t} \right| = \Omega(V^t) = \Omega(V^{\ell u+1})$. For sufficiently large V , this exceeds $V^{(\ell+o(1))u} = V^{\ell u+o(1)}$. Therefore, there exist two distinct sets $T, T' \in \binom{Q'}{t}$ satisfying $\text{Enc}(F \cup T) = \text{Enc}(F \cup T')$. The fixed decoder then gives the same output on two distinct valid inputs, leading to a contradiction. $\square$

5.3 Successful w.h.p. Requires Orientability

For $D \in \binom{\mathcal{U}^*}{d}$, let $H_{d,V}^\omega(D) := ([M_d], \{S_{d,V}^\omega(x) : x \in D\})$ be the induced multihypergraph, where parallel edges are allowed. Section 2.2 defines ℓ -orientability. We now show that orientability is necessary for recovery by an arbitrary decoder, up to a $o(1)$ term.

LEMMA 14 (KEY LEMMA: SUCCESSFUL RECOVERY REQUIRES ORIENTABILITY). *For every fixed d and every ω of the public randomness,*

$$\Pr_D \left[\text{Dec}^\omega(\text{Enc}^\omega(D)) = D \wedge H_{d,V}^\omega(D) \text{ is not } \ell\text{-orientable} \right] = o_V(1),$$

where the $o_V(1)$ term is uniform in ω .

$$\forall \omega, \quad \text{Succ}_{d,V}^\omega(\mathcal{A}) \leq \Pr_D \left[H_{d,V}^\omega(D) \text{ is } \ell\text{-orientable} \right] + o_V(1).$$

Remark. Note that the necessary condition here is *orientability* instead of *peelability*, while peelability is the sufficient condition for the particular decoder of XYZ-Sketch. The intuition of the fact can be summarized as follows: as a non-orientable hypergraph contains an overloaded "core" with $t > \ell r$ item identities confined to r cells, those cells contain only $(\ell r + o(1)) \log_2 V$ bits of information and therefore cannot distinguish all $V^{t-o(1)}$ cases.

PROOF. We will use the following equivalent way to sample D . First sample the multiplicity of each support type, i.e. $S_{d,V}^\omega(D)$, and then sample the identities within each type uniformly. Since orientability depends only on the type, once the type is fixed, the remaining identities can be sampled in any order.

Fix ω . Call a writing support $R \subseteq [M_d]$ *good* if its preimage under $S_{d,V}^\omega$ has size at least $V/\log V$. Since there are at most $N_d := \sum_{i=0}^k \binom{M_d}{i}$ possible supports, the union bound gives that uniformly random $D_{d,V}$ contains an element with a non-good writing support with probability at most $dN_d/\log V = o_V(1)$.

Condition on $S(D)$ being entirely good. If $H_{d,V}^\omega(D_{d,V})$ is not ℓ -orientable, Hall's theorem yields a submultiset $\mathcal{E}$ of t edges with vertex union $R := \bigcup_{e \in \mathcal{E}} e$ of size r such that $t > \ell r$. Choose such $\mathcal{E}$ deterministically, and first sample all the identities of elements outside $\mathcal{E}$. For each support type G , let m_G denote the multiplicity of G in $\mathcal{E}$, then the remaining identities have

$$\prod_G \binom{|S^{-1}(G) \setminus P|}{m_G} = \Omega(V^{t-o(1)})$$

possible candidates, where P is the fixed outside set. By canonicity we can insert P before the witness elements. Every witness element modifies only cells in R , so all resulting sketches agree

outside R . Hence they have at most $V^{\ell r + o_V(1)}$ possible states. We use the elementary fact that a uniformly random variable on N possibilities cannot be recovered from an observation with at most M possible values with probability exceeding M/N , even with randomized encoders and decoders. Therefore, the conditional average success probability is at most

$$\frac{N_{\text{state}}}{N_{\text{candidates}}} \leq V^{\ell r - t + o_V(1)} \leq V^{-1 + o_V(1)} = o_V(1),$$

Averaging over all possible $S(D)$ and all outside identities, and then restoring the $o_V(1)$ probability of a non-good support, proves the first claim. $\square$

5.4 Space Lower Bounds and Conditional Optimality

Fix ω , let $X_1, \dots, X_d$ be independent uniform samples from $\mathcal{U}^*$, and define

$$\tilde{H}_{d,V}^\omega := ([M_d], \{S_{d,V}^\omega(X_i) : i \in [d]\}).$$

Conditioned on $X_1, \dots, X_d$ being distinct, their underlying set is uniform over $\binom{\mathcal{U}^*}{d}$, in this case $\tilde{H}_{d,V}^\omega = H_{d,V}^\omega(\{X_1, \dots, X_d\})$; the conditioning fails with probability at most $\binom{d}{2}/(V-1) = o_V(1)$. Therefore, Lemma 14 gives

$$\text{Succ}_{d,V}^\omega(\mathcal{A}) \leq \Pr[\tilde{H}_{d,V}^\omega \text{ is } \ell\text{-orientable}] + o_V(1).$$

Theorem 3 (Barrier 2, Precise Version). For every fixed $k \geq 2$ and $\ell \geq 1$, every w.h.p. successful fixed-support canonical sketch family satisfies

$$\liminf_{d \rightarrow \infty} \frac{\mathfrak{C}_{\mathcal{A}}(d)}{d} \geq \frac{\ell}{\rho_{k,\ell}} > 1,$$

where $\rho_{k,\ell} \in (0, \ell)$ is the unique positive root¹² of equation

$$\rho_{k,\ell} = \mathbb{E}[\min\{\ell, \text{Pois}(k\rho_{k,\ell})\}]$$

PROOF. For $v \in [M_d]$, let $q_v := \Pr[v \in S_{d,V}^\omega(X_1)]$. Then $\sum_v q_v \leq k$ and $\deg_{\tilde{H}_{d,V}^\omega}(v) \sim \text{Bin}(d, q_v)$. If $\tilde{H}_{d,V}^\omega$ is ℓ -orientable, then by definition $d \leq Z$, where $Z := \sum_v \min\{\ell, \deg_{\tilde{H}_{d,V}^\omega}(v)\}$.

Let $g_{d,\ell}(q) := \mathbb{E}[\min\{\ell, \text{Bin}(d, q)\}]$. It can be easily shown to be monotone and concave in q . By Markov's inequality and Jensen's inequality,

$$\text{Succ}_{d,V}^\omega(\mathcal{A}) \leq \Pr[Z \geq d] + o(1) \leq \frac{1}{d} \sum_{v=1}^{M_d} g_{d,\ell}(q_v) + o(1) \leq \frac{M_d}{d} g_{d,\ell}\left(\frac{k}{M_d}\right) + o(1).$$

Averaging over ω gives the same bound for $\text{Succ}_{d,V}(\mathcal{A})$.

For a w.h.p. successful family $\mathcal{A}$, there must be $M_d \geq d/\ell$ for sufficiently large d , or orientability is impossible. Let $\rho_d := d/M_d \leq \ell$. Along every subsequence with $\rho_d \rightarrow \rho > 0$, Poisson convergence gives

$$\frac{M_d}{d} g_{d,\ell}\left(\frac{k}{M_d}\right) = \frac{\mathbb{E}[\min\{\ell, \text{Bin}(d, k\rho_d/d)\}]}{\rho_d} \rightarrow \frac{\mathbb{E}[\min\{\ell, \text{Pois}(k\rho)\}]}{\rho}.$$

Since $\text{Succ}_{d,V}(\mathcal{A}) \rightarrow 1$, there must be $\mathbb{E}[\min\{\ell, \text{Pois}(k\rho)\}] \geq \rho$. Thus $\limsup_d \rho_d \leq \rho_{k,\ell}$. Since $\mathfrak{C}_{\mathcal{A}}(d)/d = \ell/\rho_d$, the result follows. $\square$

For a distribution μ on $\binom{[n]}{\leq k}$, let $H_{n,m}^\mu$ be the multihypergraph obtained by sampling m independent edges from μ . We state the following conjecture:

¹²The function F on the right-hand side is increasing and strictly concave in ρ , with $F(0) = 0$, $F'(0) = k > 1$, $F(\ell) < \ell$; hence the positive root is unique.

Conjecture 15 (Uniform-Support Extremality). For every fixed $k, \ell \geq 1$ and $\varepsilon > 0$,

$$\sup_{\mu} \Pr \left[H_{n, \lceil (c_{k,\ell}^{\text{orient}} + \varepsilon)n \rceil}^{\mu} \text{ is } \ell\text{-orientable} \right] = o_n(1),$$

where the supremum ranges over all distributions on $\binom{[n]}{\leq k}$.

This conjecture formalizes and strengthens an informal conjecture of Walzer [26, Conclusion]. Under this conjecture, the following optimality result follows immediately.

Theorem 1 (Main Result, Precise Version). Assume Conjecture 15. For every fixed $k, \ell \geq 2$, every w.h.p successful fixed-support canonical sketch family $\mathcal{A}$ satisfies

$$\liminf_{d \rightarrow \infty} \frac{\mathfrak{C}_{\mathcal{A}}(d)}{d} \geq \frac{\ell}{c_{k,\ell}^{\text{orient}}} \geq \limsup_{d \rightarrow \infty} \frac{\mathfrak{C}_{XYZ}(d)}{d}.$$

PROOF. Suppose otherwise. Then, for some $\varepsilon > 0$ and infinitely many d , $\frac{d}{M_d} \geq c_{k,\ell}^{\text{orient}} + \varepsilon$ hold. By Theorem 3, $M_d = \Omega(d)$ along this subsequence. It can be noted that, for every ω and V , the hypergraph $\tilde{H}_{d,V}^{\omega}$ has distribution $H_{M_d,d}^{\mu}$, where μ is defined as the distribution of $S_{d,V}^{\omega}(X_1)$. Since orientability is preserved under deleting edges, the conjecture gives

$$\Pr \left[\tilde{H}_{d,V}^{\omega} \text{ is } \ell\text{-orientable} \right] = o_d(1),$$

where $o_d(1)$ is uniform in ω and V . Averaging the orientability bound above over ω yields

$$\text{Succ}_{d,V}(\mathcal{A}) \leq o_d(1) + o_V(1),$$

contradicting w.h.p. success. The final inequality follows from Theorem 12. $\square$

Therefore, improving upon XYZ-Sketch requires either refuting this conjecture or considering sketches outside this model. However, to the best of our knowledge, even with only the requirement $C = \text{poly}(d)$, no nontrivial unfixed-support or non-canonical $O(1)$ -update algorithm is known to succeed w.h.p.. Intuitively, substantially exploiting the unfixed-support or non-canonical conditions may make the sketch structure highly disordered, complicating the design of a decoding algorithm.

6 Experiments

We evaluate three questions: 1) whether the practical performance exhibits the predicted sharp threshold. 2) how the spatial parameters affect the peeling process, and 3) what end-to-end time-space tradeoff is obtained in the implementation of XYZ-Sketch.

6.1 Experimental Setup

We generate two sets $A, B \subseteq \mathcal{U}^*$ with $|(A \setminus B) \cup (B \setminus A)| = d$, split the difference nearly evenly between the two directions, and randomly shuffle their insertion orders. Unless otherwise stated, $\mathcal{U} = \mathbb{F}_{998244353}$, $w := \lceil \log_2 |\mathcal{U}| \rceil = 30$ and $|A| = |B| = 10^7$. We normalize communication by

$$\mathfrak{R} := \frac{\text{serialized sketch size in bits}}{dw},$$

A trial succeeds only if the exact signed symmetric difference is recovered. Probability experiments use 100 independent trials, target success probability 0.9.

Figures 1(a) and 2 use the complete XYZ-Sketch C++ implementations, including murmur hash, D-RFR recovery, verification, and decoding. Figure 1(b)–(c) instead uses an ideal-cell hypergraph simulator that reproduces the circular placement and peeling process, but recovers a cell exactly whenever its residual degree is at most ℓ . It avoids the randomness unrelated to the hypergraph

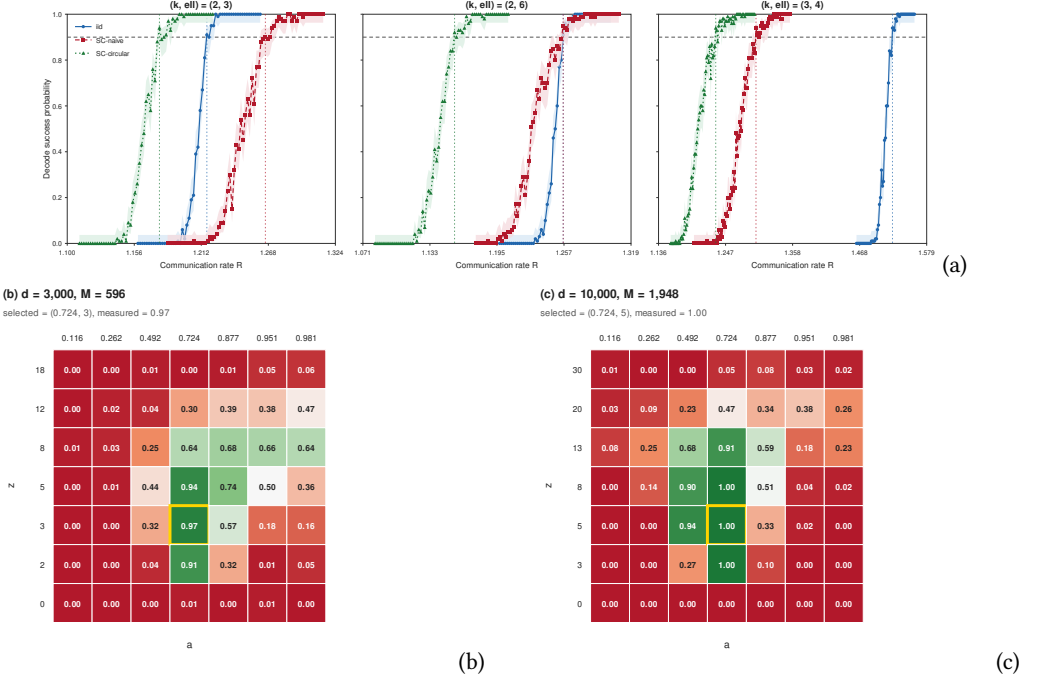


Fig. 1. Structural behavior of XYZ-Sketch. (a) Full-implementation success against normalized communication, where $d = 10^4$ and (a, z) follows our heuristic choice. The Wilson 95% CI is shown as a light-colored band. (b) Ideal-cell peeling success over circular parameters (a, z) of $d = 3,000$, $M = 596$ and $(k, \ell) = (2, 6)$. The yellow box marks the heuristic parameter choice. (c) Another peeling-success heatmap at $d = 10,000$ and $M = 1,948$.

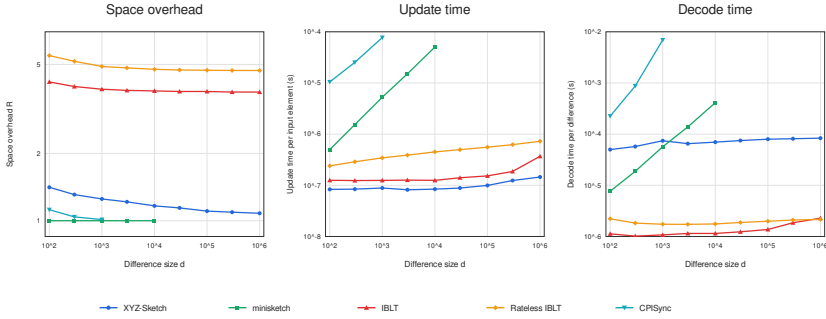


Fig. 2. End-to-end tradeoffs at a 90% success target: (a) normalized space cost R ; (b) update time per element; (c) decode time per element.

structure introduced by pure cell verification, thereby placing greater emphasis on the hypergraph structure itself.

All end-to-end algorithms are evaluated on shared datasets. Experiments run on a server with 18-core processor and 128 GB DRAM memory.

6.2 Thresholds and Spatial Parameters

We first calibrate the two constants in our heuristic using only smaller-scale instances, obtaining $C \approx 0.875$ and $D \approx 1.19$, and then keep them fixed for the remaining experiments. Figure 1(a) shows sharp success transitions for several representative (k, ℓ) pairs. Circular spatial coupling consistently improves over both i.i.d. placement and naive coupling, matching the predicted threshold gain. We further examine $(k, \ell) = (2, 6)$ as a representative case. Figures 1(b)–(c) show that the resulting choices of (a, z) lie inside the high-success regions and continue to perform well when extrapolated to substantially larger values of d and M . A more comprehensive collection of heatmaps is provided in Appendix Figure 3. Together, these results support the robustness of our heuristic parameter rule and are consistent with the threshold-saturation behavior of circular spatial coupling.

6.3 End-to-End Tradeoffs

Figure 2 compares XYZ-Sketch with prior works (See Appendix D for details) under the same success target 0.9. One can see that our XYZ-Sketch has an insertion efficiency close to IBLT, while the communication cost rapidly approaches minisketch.

The drawback of XYZ-Sketch is its decoding efficiency, which does not match the superior performance of IBLT but is still acceptable. Since it is mainly caused by the root-finding process in a single cell, we believe that employing a Minisketch-like approach for root-finding, such as generating lookup tables at runtime, coupled with further hardware-assisted engineering optimizations, can significantly accelerate the decoding process. However, this falls outside the scope of this work.

7 Conclusion

We studied the trade-off between space and update time in the set reconciliation problem, especially for the streaming setting. XYZ-Sketch replaces the singleton peeling rule of conventional IBLTs with high-capacity cells which is realized by D-RFR, and combines them with spatial coupling. For any constants $\epsilon, \delta > 0$, suitable parameters allow XYZ-Sketch to reconcile sets with difference at most d using $\mathfrak{C} \leq (1 + \epsilon)d$, $O(1)$ update time, expected $O(d \log V)$ decoding time, and failure probability at most δ for sufficiently large d . Our experiments further demonstrate the predicted near-optimal performance in both metrics. We further formulated a fixed-support canonical model, showed that, under the Uniform-Support Extremality Conjecture, XYZ-Sketch asymptotically reaches the frontier in this model. The unconditional zero-error and space-lower-bound barriers are also established within this model.

An important open question is to prove or refute Conjecture 15, thereby determining whether the conditional frontier is exact. It also remains to understand how the frontier changes beyond the fixed-support canonical model. On the practical side, it would be interesting to extend high-capacity cells to multi-party and rateless set reconciliation, and also to explore their use in other sketching and probabilistic data structures.

8 Methods: Use and Disclosure of AI and LLMs

The authors developed the core idea, proof outline, and C++ implementation of the core algorithm. Generative AI assisted with writing, literature review, citation checking, refining proof details for specific lemmas, and developing evaluation/plotting scripts; all claims and outputs were independently reviewed and verified by the authors.

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Appendix

The appendix is organized as follows. Appendix A collects the notation used throughout the paper and the relevant peelability and orientability threshold values. Appendix B provides the previously missing proofs. Appendix C presents the complete D-RFR reconstruction procedure, including its reduction to the classical RFR problem and the implementation details. Appendix D provides the Supplementary Experimental Figures. Appendix E provides the full pseudocode of XYZ-Sketch.

A Tables

A.1 Notation Table

Table 2 lists only the global notation reused across sections. Symbols introduced and discharged within a single proof or algorithm are defined locally and omitted here.

Table 2. Core notation used throughout the paper.

Symbol	Meaning
Problem and hypergraph model	
$\mathcal{U} = \mathbb{F}_p, V = p$	Universe and its cardinality; $\mathcal{U}^* = \mathcal{U} \setminus \{0\}$.
A, B, d	The two input sets and the known upper bound $ (A \setminus B) \cup (B \setminus A) \leq d$.
$\mathfrak{C}(d)$	Communication in $\mathcal{U}$ -word equivalents.
ε, δ	Target communication overhead and failure probability.
$H = (\mathcal{V}, \mathcal{E})$	A multihypergraph with vertex set $\mathcal{V}$ and edge multiset $\mathcal{E}$.
k, ℓ	Number of cells touched per element and local recovery/peeling capacity.
$H_{n,m}^{(k)}, c = m/n$	Fully random k -uniform hypergraph and its edge density.
$c_{k,\ell}^{\text{peel}}, c_{k,\ell}^{\text{orient}}$	Thresholds for ℓ -peelability and ℓ -orientability.
$Q(\lambda, r)$	Poisson upper tail $\Pr[\text{Pois}(\lambda) \geq r]$.
XYZ-Sketch and the high-capacity cell	
$M, \mathcal{B}, \mathcal{D}$	Number of cells, one encoded sketch, and the residual sketch after subtraction.
$h_1, \dots, h_k, \Gamma(x)$	Placement hash functions and the deduplicated support of element x .
C, f_C	A cell state and the signed multiplicity map represented by it.
$c_C, P_C(Z), \text{fp}_C$	Signed count, truncated polynomial, and optional fingerprint stored in cell C .
h_{fp}, q	Independent fingerprint hash and its odd modulus.
$Z, \chi_S(Z)$	Formal polynomial variable and the characteristic polynomial of set S .
$R(Z), F(Z), G(Z), m_C$	D-RFR residue, reconstructed numerator/denominator, and their signed degree difference.
Spatial coupling and finite-size parameters	
z	Coupling-length parameter.
a	Circular parameter; $a = 0$ is the original version.
$g_0, g_1, \dots, g_k, h'_i, h_i^{(a)}$	Anchor/window randomness and the terminated/circular placement rules.
$a_{k,\ell}, z_{k,\ell}$	Threshold-informed finite-size design choices for a and z .
Barrier model	
$\mathcal{A}, \omega$	A fixed-support canonical sketch family and public randomness.
$M_d, \mathfrak{C}_{\mathcal{A}}(d) = \ell M_d$	Persistent cell count and word-scale state of $\mathcal{A}$.
$S_{d,V}^\omega(x)$	Predetermined write support of element x .
Enc, Upd, Dec	Encoding, update, and decoding maps (with indices shown when needed).

Table 2, continued.

Symbol	Meaning
$\text{Succ}_{d,V}(\mathcal{A})$	Success probability, averaged over public randomness.
$H_{d,V}^\omega(D), \tilde{H}_{d,V}^\omega$	Support hypergraphs induced by sampling without and with replacement.
$\rho_d = d/M_d, \rho_{k,\ell}$	Induced density and the positive root of $\rho = \mathbb{E}[\min\{\ell, \text{Pois}(k\rho)\}]$.
$\mu, H_{n,m}^\mu$	A distribution on supports and the resulting i.i.d.-edge multihypergraph.

A.2 Threshold Table

For $r \geq 1$, define

$$Q(\xi, r) = \Pr[\text{Pois}(\xi) \geq r].$$

Except for $(k, \ell) = (2, 1)$, the peeling threshold is

$$c_{k,\ell}^{\text{peel}} = \min_{\xi > 0} \frac{\xi}{kQ(\xi, \ell)^{k-1}},$$

where the minimizing ξ is the positive solution of

$$Q(\xi, \ell) = (k-1)\xi \Pr[\text{Pois}(\xi) = \ell-1].$$

For orientability, let ξ^* solve

$$k\ell = \xi^* \frac{Q(\xi^*, \ell)}{Q(\xi^*, \ell+1)}.$$

Then

$$c_{k,\ell}^{\text{orient}} = \frac{\xi^*}{kQ(\xi^*, \ell)^{k-1}}.$$

The limiting special case is $c_{2,1}^{\text{peel}} = 0$ and $c_{2,1}^{\text{orient}} = 1/2$. These formulas and the threshold interpretation follow the random-hypergraph results used by the paper [10].

Since $M \simeq d/c$ cells are required at density c , and each cell stores ℓ field words, the asymptotic word-space ratio is

$$\frac{\mathfrak{C}}{d} \simeq \frac{\ell M}{d} = \frac{\ell}{c}.$$

Table 4 gives the corresponding reciprocals of the normalized threshold efficiencies.

For interpreting communication, a capacity- ℓ cell has asymptotic word-overhead ℓ/c . Table 5 gives the tuples used most often in the experiment plan. The ratio $c^{\text{peel}}/c^{\text{orient}}$ is also the uncalibrated part of the proposed seed rule $a_{k,\ell} = Cc_{k,\ell}^{\text{peel}}/c_{k,\ell}^{\text{orient}}$.

Table 3. Thresholds for the fully random k -uniform ensemble. Each entry is $c_{k,\ell}^{\text{peel}}/c_{k,\ell}^{\text{orient}}$ under the paper's density convention $c = m/n$. Values are rounded to four decimal places.

$\ell \backslash k$	2	3	4	5	6	7
1	0.0000/0.5000	0.8185/0.9179	0.7723/0.9768	0.7018/0.9924	0.6371/0.9974	0.5818/0.9991
2	1.6755/1.7940	1.5528/1.9764	1.3336/1.9965	1.1578/1.9994	1.0216/1.9999	0.9146/2.0000
3	2.5747/2.8775	2.1745/2.9919	1.8109/2.9994	1.5457/3.0000	1.3488/3.0000	1.1977/3.0000
4	3.3996/3.9215	2.7467/3.9970	2.2498/3.9999	1.9022/4.0000	1.6492/4.0000	1.4574/4.0000
5	4.1827/4.9478	3.2894/4.9989	2.6654/5.0000	2.2395/5.0000	1.9333/5.0000	1.7030/5.0000
6	4.9376/5.9644	3.8117/5.9996	3.0651/6.0000	2.5635/6.0000	2.2060/6.0000	1.9386/6.0000
7	5.6721/6.9754	4.3189/6.9998	3.4528/7.0000	2.8776/7.0000	2.4703/7.0000	2.1668/7.0000
8	6.3905/7.9828	4.8143/7.9999	3.8312/8.0000	3.1840/8.0000	2.7279/8.0000	2.3892/8.0000

Table 4. Actual asymptotic word-space ratios at the fully random peeling and orientability thresholds. Each entry is $(\mathfrak{C}/d)_{\text{peel}}/(\mathfrak{C}/d)_{\text{orient}} = (\ell/c_{k,\ell}^{\text{peel}})/(\ell/c_{k,\ell}^{\text{orient}})$. The value ∞ for $(k, \ell) = (2, 1)$ reflects $c_{2,1}^{\text{peel}} = 0$.

$\ell \backslash k$	2	3	4	5	6	7
1	∞ /2.0000	1.2218/1.0894	1.2949/1.0238	1.4249/1.0076	1.5697/1.0026	1.7189/1.0009
2	1.1937/1.1148	1.2880/1.0119	1.4997/1.0018	1.7275/1.0003	1.9577/1.0000	2.1868/1.0000
3	1.1652/1.0426	1.3796/1.0027	1.6567/1.0002	1.9409/1.0000	2.2242/1.0000	2.5049/1.0000
4	1.1766/1.0200	1.4563/1.0007	1.7780/1.0000	2.1029/1.0000	2.4254/1.0000	2.7445/1.0000
5	1.1954/1.0106	1.5201/1.0002	1.8759/1.0000	2.2327/1.0000	2.5863/1.0000	2.9361/1.0000
6	1.2152/1.0060	1.5741/1.0001	1.9575/1.0000	2.3406/1.0000	2.7199/1.0000	3.0951/1.0000
7	1.2341/1.0035	1.6208/1.0000	2.0273/1.0000	2.4326/1.0000	2.8337/1.0000	3.2306/1.0000
8	1.2518/1.0022	1.6617/1.0000	2.0881/1.0000	2.5126/1.0000	2.9327/1.0000	3.3484/1.0000

Table 5. Derived quantities for the principal operating points.

k	ℓ	c^{peel}	c^{orient}	$c^{\text{peel}}/c^{\text{orient}}$	ℓ/c^{peel}	ℓ/c^{orient}
3	1	0.8185	0.9179	0.8916	1.2218	1.0894
2	3	2.5747	2.8775	0.8948	1.1652	1.0426
2	6	4.9376	5.9644	0.8278	1.2152	1.0060
3	4	2.7467	3.9970	0.6872	1.4563	1.0007

B The Missing Proof in Section 4

Preliminary knowledge. We first recall some simple facts used below.

1) If $F(0) \neq 0$, then F is invertible modulo Z^ℓ , so $F'(Z)/F(Z) \bmod Z^\ell$ is well defined. Moreover, if $F(Z) \equiv 1 \pmod{Z^\ell}$, then $F'(Z)/F(Z) \equiv 0 \pmod{Z^{\ell-1}}$.

2) For every $a \in \mathcal{U}^* = \mathbb{F}_p^*$, we have $\frac{1}{Z-a} \equiv -\sum_{r \geq 1} a^{-r} Z^{r-1} \pmod{Z^\ell}$.

3) If m polynomials in m variables have degrees $D_1, \dots, D_m$, then they have at most $\prod_{i=1}^m D_i$ isolated common zeros over the algebraic closure. A common zero at which the Jacobian has full rank is *isolated*. The proposition is a standard consequence of the Bézout theorem in algebraic geometry.

LEMMA 16. *Let $p > \max\{\ell, 2\}$ be prime. Fix $e_1, \dots, e_\ell \in \{-2, -1, 1, 2\}$, $c_0 \in \mathbb{F}_p \setminus \{0\}$, and $c_1, \dots, c_{\ell-1} \in \mathbb{F}_p$. Then the system*

$$\prod_{i=1}^{\ell} y_i^{-e_i} = c_0; \quad \sum_{i=1}^{\ell} e_i y_i^r = c_r, \quad r = 1, \dots, \ell - 1$$

has at most $2\ell(\ell-1)! = O_\ell(1)$ solutions $(y_1, \dots, y_\ell) \in (\overline{\mathbb{F}}_p \setminus \{0\})^\ell$ in which the y_i are pairwise distinct.

PROOF. Let $P(y) := \prod_{e_i > 0} y_i^{e_i}$, $Q(y) := \prod_{e_i < 0} y_i^{-e_i}$. The equation can be written as

$$H_0(y_1, \dots, y_\ell) := Q(y) - c_0 P(y) = 0$$

$$H_r(y_1, \dots, y_\ell) := \sum_{i=1}^{\ell} e_i y_i^r - c_r = 0, \quad r = 1, \dots, \ell - 1,$$

Consider a solution in which the y_i are nonzero and pairwise distinct. Since $H_0(y) = 0$, we have $Q(y) = c_0 P(y) \neq 0$. The Jacobian matrix of H at this solution is

$$\begin{pmatrix} -\frac{e_1 Q(y)}{y_1} & \dots & -\frac{e_\ell Q(y)}{y_\ell} \\ e_1 & \dots & e_\ell \\ 2e_1 y_1 & \dots & 2e_\ell y_\ell \\ \vdots & & \vdots \\ (\ell-1)e_1 y_1^{\ell-2} & \dots & (\ell-1)e_\ell y_\ell^{\ell-2} \end{pmatrix}.$$

Multiply the i -th column of the Jacobian by y_i/e_i . Then multiply the first row by $-Q(y)^{-1}$ and, for each $r = 1, \dots, \ell - 1$, multiply the row corresponding to the r -th equation by r^{-1} . All these scalars are nonzero. Each operation multiplies the determinant by a nonzero scalar and therefore preserves whether the determinant is zero. The resulting matrix is

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ y_1 & y_2 & \dots & y_\ell \\ \vdots & \vdots & & \vdots \\ y_1^{\ell-1} & y_2^{\ell-1} & \dots & y_\ell^{\ell-1} \end{pmatrix}.$$

Its determinant is $\prod_{1 \leq i < j \leq \ell} (y_j - y_i) \neq 0$. Hence, the original Jacobian has full rank and very such solution is isolated.

Also, we have

$$\deg H_0 \leq \max\{\deg P, \deg Q\} \leq 2\ell; \quad \forall r \geq 1, \deg H_r = r.$$

Therefore, the number of such solutions is at most $2\ell(\ell-1)!$. The polynomial system may have additional solutions with some $y_i = 0$, but this does not affect the bound of the isolated solutions under consideration. $\square$

MISSING PROOF OF LEMMA 7. Let $f \neq g$ be the two fixed signed maps in the lemma. Define $e(x) := f(x) - g(x)$ and $J := \{x : e(x) \neq 0\}$, and let $s := |J|$. For every $x \in J$, we have $e(x) \in \{-2, -1, 1, 2\}$.

Suppose that their polynomial states agree. Then

$$R(Z) := \prod_{x \in J} (Z - \phi(x))^{e(x)} \equiv 1 \pmod{Z^\ell}.$$

Since every $\phi(x)$ is nonzero, all factors are invertible modulo Z^ℓ . We have

$$\frac{R'(Z)}{R(Z)} = \sum_{x \in J} \frac{e(x)}{Z - \phi(x)} \equiv 0 \pmod{Z^{\ell-1}}.$$

Expanding each fraction at $Z = 0$, we obtain

$$\sum_{x \in J} e(x) \phi(x)^{-r} = 0, \quad r = 1, \dots, \ell - 1.$$

We first show that $s \geq \ell$. Otherwise, write $J = \{x_1, \dots, x_s\}$ and let $y_i := \phi(x_i)^{-1}$. The first s equations above give

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_s \\ y_1^2 & y_2^2 & \cdots & y_s^2 \\ \vdots & \vdots & & \vdots \\ y_1^s & y_2^s & \cdots & y_s^s \end{pmatrix} \begin{pmatrix} e(x_1) \\ e(x_2) \\ \vdots \\ e(x_s) \end{pmatrix} = 0.$$

As the determinant of the Vandermonde matrix is $(\prod_{i=1}^s y_i) \prod_{1 \leq i < j \leq s} (y_j - y_i)$, which is nonzero because ϕ is a bijection into $\mathcal{U}^*$. Hence every $e(x_i)$ is zero, contradicting the definition of J . Therefore, $s \geq \ell$.

Choose distinct $x_1, \dots, x_\ell \in J$ and fix the labels ϕ of all elements in $J \setminus \{x_1, \dots, x_\ell\}$. Let $e_i := e(x_i)$ and $y_i := \phi(x_i)^{-1}$. The constant term of $R(Z) \equiv 1 \pmod{Z^\ell}$, together with the equations above, implies the system

$$\prod_{i=1}^{\ell} y_i^{-e_i} = c_0; \quad \sum_{i=1}^{\ell} e_i y_i^r = c_r, \quad r = 1, \dots, \ell - 1$$

where $c_0, \dots, c_{\ell-1}$ are fixed after conditioning.

Lemma 16 implies that this system of equations has only $O_\ell(1)$ solutions in which the y_i are nonzero and pairwise distinct. Finally, after fixing the other $s - \ell$ labels, the ordered tuple $(\phi(x_1), \dots, \phi(x_\ell))$ is sampled uniformly without replacement from the remaining $N - s + \ell$ labels. As $s \leq d + \ell$ and $d \leq N/2$, we have $N - s + \ell \geq N/2$. The number of possible ordered tuples is hence $\binom{N-s+\ell}{\ell} \ell! = \Omega_\ell(N^\ell)$. Moreover, the map $\phi(x_i) \mapsto y_i = \phi(x_i)^{-1}$ is a bijection and preserves distinctness. Therefore, for every fixing of the other $s - \ell$ labels,

$$\Pr_{\phi} [\Psi_f(Z) \equiv \Psi_g(Z) \pmod{Z^\ell} \mid \text{the fixed labels}] = O_\ell(N^{-\ell}).$$

Averaging over the fixed labels gives

$$\Pr_{\phi} [\Psi_f(Z) \equiv \Psi_g(Z) \pmod{Z^\ell}] = O(N^{-\ell}).$$

□

C Implementation of the D-RFR Procedure

We now present the reconstruction procedure for D-RFR in Section 3.2. As the uniqueness part has been proved in the main text, this appendix focuses on how to recover the desired pair of polynomials.

C.1 The Original RFR

Next, we illustrate how to *find a solution* through a series of lemmas, building the argument step by step. We first introduce the original RFR as follows.

LEMMA 17 (THE ORIGINAL RFR ALGORITHM [7, 15]). *Let $N = 2D + 1$ be odd, and let $R(Z)$ be a polynomial with $\deg R < N$. Suppose that there exist coprime polynomials $F, G \in \mathcal{U}[Z]$ such that*

- $\deg F \leq D, \deg G \leq D$, and $G(0) \neq 0$;
- $\frac{F(Z)}{G(Z)} \equiv R(Z) \pmod{Z^N}$.

Then the original RFR algorithm recovers F and G , up to multiplication by a common nonzero constant, in $O(N \log^2 N)$ field operations. In particular, if G is known to be monic, we can recover the required normalization.

PROOF. If $R = 0$, the coprimality and degree assumptions imply that the reduced solution is $(F, G) = (0, 1)$, so we assume $R \neq 0$ below. One can see that

$$\begin{aligned} \frac{F(Z)}{G(Z)} &\equiv R(Z) \pmod{Z^N} \\ \iff \exists K \in \mathcal{U}[Z]: \quad G(Z)R(Z) + K(Z)Z^N &= F(Z). \end{aligned}$$

Thus, our goal is to find $G(Z)$ and $K(Z)$ such that $\deg G \leq D$ and $\deg(G(Z)R(Z) + K(Z)Z^N) \leq D$.

Denoting $A := Z^N$ and $B := R(Z)$, we use an approach inspired by the Euclidean algorithm and formulate a recurrence relation as follows:

$$\begin{aligned} H_0(Z) &:= A, \quad H_1(Z) := B, \\ H_i(Z) &:= H_{i-2}(Z) \bmod H_{i-1}(Z), \quad i \geq 2, \\ Q_i(Z) &:= \frac{H_{i-2}(Z) - H_i(Z)}{H_{i-1}(Z)}. \end{aligned}$$

We continue the definition until $H_i(Z) = 0$. Since $\mathcal{U}[Z]$ is a Euclidean domain, the process must end. It is easy to prove that

- $\deg H_i < \deg H_{i-1}$;
- $Q_i \in \mathcal{U}[Z]$ and $\deg Q_i = \deg H_{i-2} - \deg H_{i-1}$.

We define

$$\mathbf{G}_i := \begin{bmatrix} H_{i-1} \\ H_i \end{bmatrix} \in M_{2 \times 1}(\mathcal{U}[Z]).$$

Then

$$\mathbf{G}_i = \begin{bmatrix} 0 & 1 \\ 1 & -Q_i(Z) \end{bmatrix} \mathbf{G}_{i-1}.$$

For convenience, denote $\begin{bmatrix} 0 & 1 \\ 1 & -Q_i(Z) \end{bmatrix}$ by $\varphi(Q_i)$. We find p such that

$$\deg H_{p-1} > D, \quad \deg H_p \leq D.$$

Then

$$\mathbf{G}_p = \varphi(Q_p)\varphi(Q_{p-1}) \cdots \varphi(Q_2) \begin{bmatrix} A \\ B \end{bmatrix}.$$

We denote the matrix $\varphi(Q_p) \cdots \varphi(Q_2)$ by $\mathcal{H}_N(A, B)$. More generally, when the subscript is an arbitrary integer t , $\mathcal{H}_t$ denotes the same Euclidean prefix with the stopping threshold $\lfloor t/2 \rfloor$. For every entry $x(Z)$ in this matrix,

$$\begin{aligned} \deg x &\leq \sum_{i=2}^p \deg Q_i = \deg H_0 - \deg H_{p-1} \\ &\leq N - (D + 1) = D. \end{aligned}$$

Suppose that

$$\mathcal{H}_N(A, B) = \begin{bmatrix} x_0(Z) & y_0(Z) \\ x_1(Z) & y_1(Z) \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} H_{p-1}(Z) \\ H_p(Z) \end{bmatrix} = \begin{bmatrix} x_0(Z) & y_0(Z) \\ x_1(Z) & y_1(Z) \end{bmatrix} \begin{bmatrix} Z^N \\ R(Z) \end{bmatrix},$$

and hence

$$H_p(Z) = x_1(Z)Z^N + y_1(Z)R(Z).$$

Since $\deg H_p, \deg x_1, \deg y_1 \leq D$, the pair (H_p, y_1) satisfies the required degree bounds and the polynomial congruence

$$H_p(Z) \equiv y_1(Z)R(Z) \pmod{Z^N}.$$

It remains to explain why this candidate yields the desired reduced pair. Combining the last congruence with $F \equiv GR \pmod{Z^N}$ gives

$$H_p G \equiv y_1 F \pmod{Z^N}.$$

Both sides have degree at most $2D = N - 1$, so in fact $H_p G = y_1 F$. Since $\gcd(F, G) = 1$, there is a polynomial C such that $(H_p, y_1) = C(F, G)$. Dividing H_p and y_1 by their monic greatest common divisor therefore recovers (F, G) up to a common nonzero constant. We finally normalize the component that is known to be monic.

Next, we illustrate how to efficiently compute $\mathcal{H}_N(A, B)$ for $A, B \in \mathcal{U}[Z]$ with $\deg A, \deg B \leq N$, omitting the standard base cases and leading-coefficient normalizations for brevity. For an integer $L > 0$, divide A, B as

$$\begin{bmatrix} A(Z) \\ B(Z) \end{bmatrix} = Z^L \begin{bmatrix} A_1(Z) \\ B_1(Z) \end{bmatrix} + \begin{bmatrix} A_2(Z) \\ B_2(Z) \end{bmatrix}, \quad \max\{\deg A_2, \deg B_2\} < L.$$

We suppose $\deg A_1, \deg B_1 \leq \lfloor N/2 \rfloor$. Let

$$M := \mathcal{H}_{\lfloor N/2 \rfloor}(A_1, B_1).$$

Then

$$\begin{aligned} M \begin{bmatrix} A(Z) \\ B(Z) \end{bmatrix} &= M \begin{bmatrix} Z^L A_1(Z) + A_2(Z) \\ Z^L B_1(Z) + B_2(Z) \end{bmatrix} \\ &= M \begin{bmatrix} A_1(Z) & A_2(Z) \\ B_1(Z) & B_2(Z) \end{bmatrix} \begin{bmatrix} Z^L \\ 1 \end{bmatrix}. \end{aligned}$$

Denote

$$M \begin{bmatrix} A_1(Z) & A_2(Z) \\ B_1(Z) & B_2(Z) \end{bmatrix} =: \begin{bmatrix} C_1(Z) & C_2(Z) \\ D_1(Z) & D_2(Z) \end{bmatrix}.$$

By definition,

$$M \begin{bmatrix} A_1(Z) \\ B_1(Z) \end{bmatrix} = \begin{bmatrix} C_1(Z) \\ D_1(Z) \end{bmatrix}.$$

Thus the vector $[C_1 \ D_1]^\top$ is the stopping pair in the first recursive half, leading to

$$\deg C_1 > \lfloor N/4 \rfloor \geq \deg D_1.$$

Since

$$\deg C_2, \deg D_2 \leq \lfloor N/4 \rfloor + \max\{\deg A_2, \deg B_2\} < \lfloor N/4 \rfloor + L,$$

we obtain

$$M \begin{bmatrix} A(Z) \\ B(Z) \end{bmatrix} = \begin{bmatrix} C_1(Z)Z^L + C_2(Z) \\ D_1(Z)Z^L + D_2(Z) \end{bmatrix},$$

and

$$\deg(C_1Z^L + C_2) > \lfloor N/4 \rfloor + L \geq \deg(D_1Z^L + D_2).$$

The half-GCD computation of $\mathcal{H}_N(A, B)$ can therefore be expressed as follows:

- (1) Set $L := \lceil N/2 \rceil$ and recursively compute $M_1 = \mathcal{H}_{\lfloor N/2 \rfloor}(A_1, B_1)$. Then assign

$$\begin{bmatrix} A(Z) \\ B(Z) \end{bmatrix} \leftarrow M_1 \begin{bmatrix} A(Z) \\ B(Z) \end{bmatrix}.$$

The new $B(Z)$ is the polynomial $D_1(Z)Z^L + D_2(Z)$ discussed above, hence

$$\deg B \leq \lfloor N/4 \rfloor + \lfloor N/2 \rfloor \leq \lfloor 3N/4 \rfloor.$$

- (2) Perform one Euclidean step, namely

$$\begin{bmatrix} A(Z) \\ B(Z) \end{bmatrix} \leftarrow \begin{bmatrix} B(Z) \\ A(Z) \bmod B(Z) \end{bmatrix}.$$

At this point, $\deg A, \deg B \leq \lfloor 3N/4 \rfloor$.

- (3) Set $L := \lceil N/4 \rceil$, take the corresponding high parts of the transformed pair, and recursively compute the remaining half-GCD matrix M_2 . Applying M_2 to the full transformed pair finishes the desired Euclidean prefix.
- (4) Return $M_2\varphi(Q)M_1$, where Q is the quotient in the Euclidean step above.

Each recursive level performs two half-size recursive calls and $O(N \log N)$ additional work for polynomial and matrix arithmetic. Therefore,

$$T(N) = 2T(N/2) + O(N \log N) = O(N \log^2 N).$$

□

C.2 A Shifted Form of RFR

Next, we use Lemma 17 as a tool to construct the desired algorithm. The following shifted form keeps the interface of the original balanced RFR while allowing different degree bounds.

LEMMA 18 (SHIFTED RFR). *Let $S(Z)$ be a polynomial with $\deg S < n$, and let $D, s \geq 0$ satisfy $D \geq s$ and*

$$2D + 1 \leq n + s.$$

Suppose that there exist coprime polynomials $A, B \in \mathcal{U}[Z]$ satisfying

$$B(0) \neq 0, \quad \deg A \leq D - s, \quad \deg B \leq D,$$

and

$$\frac{A(Z)}{B(Z)} \equiv S(Z) \pmod{Z^n}.$$

Then (A, B) can be recovered, up to a common nonzero scalar, in $O((n + s) \log^2(n + s))$ field operations.

PROOF. From the known residue $S(Z) \bmod Z^n$, form

$$\widetilde{S}(Z) := Z^s S(Z) \bmod Z^{2D+1}.$$

This is well-defined from the available coefficients because $2D + 1 \leq n + s$. Moreover,

$$\frac{Z^s A(Z)}{B(Z)} \equiv \widetilde{S}(Z) \pmod{Z^{2D+1}},$$

where both $Z^s A$ and B have degree at most D . Since $B(0) \neq 0$ and $\gcd(A, B) = 1$, we also have $\gcd(Z^s A, B) = 1$. Lemma 17 therefore recovers $(Z^s A, B)$ up to a common scalar. Exact division of the first component by Z^s yields A . $\square$

C.3 Degree-Aware Reconstruction

We now prove the reconstruction part of the D-RFR theorem. The reduction below first removes any power of Z from the numerator. This is needed because the theorem assumes only $G(0) \neq 0$; it does not assume $F(0) \neq 0$.

LEMMA 19 (D-RFR RECONSTRUCTION). *Let $R(Z)$ be a polynomial with $\deg R < \ell$, and let $m \in [-\ell, \ell]$. Suppose that there are coprime and monic polynomials $F(Z), G(Z)$ with*

$$G(0) \neq 0, \quad \deg F + \deg G \leq \ell, \quad \deg F - \deg G = m,$$

such that

$$\frac{F(Z)}{G(Z)} \bmod Z^\ell = R(Z).$$

Then $F(Z), G(Z)$ can be found using $R(Z)$ and m within $O(\ell \log^2 \ell)$ field operations.

PROOF. We first remove a possible power of Z . If $R = 0$, then $F \equiv 0 \pmod{Z^\ell}$ because $G(0) \neq 0$. The degree bound and monicity force $F = Z^\ell$ and $G = 1$; in particular, this case can occur only when $m = \ell$.

Suppose now that $R \neq 0$, and let

$$v := \min\{j : [Z^j]R(Z) \neq 0\}.$$

Since $G(0) \neq 0$, the congruence implies $Z^v \mid F$. Write

$$F(Z) = Z^v F_0(Z), \quad R(Z) = Z^v R_0(Z),$$

and set

$$n := \ell - v, \quad m_0 := m - v.$$

Then $F_0(0) \neq 0, G(0) \neq 0$, and

$$\frac{F_0(Z)}{G(Z)} \equiv R_0(Z) \pmod{Z^n}, \quad \deg F_0 + \deg G \leq n, \quad \deg F_0 - \deg G = m_0.$$

If $m_0 < 0$, then R_0 is invertible modulo Z^n . We replace R_0 by $R_0^{-1} \bmod Z^n$, replace m_0 by $-m_0$, and swap the roles of F_0 and G . Thus, it remains to describe the construction for $m_0 \geq 0$; after that construction, we undo this swap if it was made.

For the case $m_0 = 0$, if $R_0 = 1$, coprimality and monicity imply $F_0 = G = 1$, and we return this pair directly. Otherwise, define

$$H(Z) := F_0(Z) - G(Z).$$

Let $D := \lfloor n/2 \rfloor$. Since F_0 and G are monic and have the same degree, their leading terms cancel. Under the existence assumption, $R_0 \neq 1$ also implies $n \geq 2$, so $D \geq 1$. Consequently,

$$\deg H \leq D - 1, \quad \deg G \leq D,$$

while

$$\frac{H(Z)}{G(Z)} \equiv R_0(Z) - 1 \pmod{Z^n}.$$

Also, $\gcd(H, G) = \gcd(F_0, G) = 1$. Applying Lemma 18 with $S = R_0 - 1$ and $s = 1$ recovers (H, G) up to a common scalar; the condition $2D + 1 \leq n + 1$ holds automatically. We normalize the second component so that G is monic and return

$$(F_0, G) = (H + G, G).$$

It remains to handle $m_0 \geq 1$. Let

$$d_G := \left\lfloor \frac{n - m_0}{2} \right\rfloor, \quad D := d_G + m_0 - 1, \quad s := m_0 - 1,$$

and define

$$H(Z) := F_0(Z) - Z^{m_0}G(Z).$$

Since F_0 and $Z^{m_0}G$ are monic of the same degree, their leading terms cancel. Thus,

$$\deg G \leq d_G = D - s, \quad \deg H \leq D.$$

Furthermore, $H(0) = F_0(0) \neq 0$, $\gcd(G, H) = \gcd(G, F_0) = 1$, and

$$\frac{G(Z)}{H(Z)} \equiv \frac{1}{R_0(Z) - Z^{m_0}} \pmod{Z^n}.$$

The inverse exists because $(R_0 - Z^{m_0})(0) = R_0(0) \neq 0$. Finally,

$$\begin{aligned} 2D + 1 &= 2 \left\lfloor \frac{n - m_0}{2} \right\rfloor + 2m_0 - 1 \\ &\leq n + m_0 - 1 = n + s. \end{aligned}$$

Lemma 18, applied to

$$S(Z) := (R_0(Z) - Z^{m_0})^{-1} \pmod{Z^n},$$

therefore recovers the projective pair (G, H) . Here the normalization must be taken from the *first* component: we scale both components so that G is monic. We then return

$$(F_0, G) = (H + Z^{m_0}G, G).$$

Finally, if the reciprocal reduction was used, we swap the two reconstructed polynomials back. We then restore the removed power of Z and output

$$(F, G) = (Z^v F_0, G).$$

□

C.4 Discussion on Practical Implementation

The theorem above assumes that a valid pair exists. On an arbitrary decoder input, D-RFR accepts a reconstructed candidate only after the following deterministic checks:

- F and G are monic;
- $G(0) \neq 0$ and $\gcd(F, G) = 1$;
- $\deg F - \deg G = m$ and $\deg F + \deg G \leq \ell$;
- $F(Z) \equiv G(Z)R(Z) \pmod{Z^\ell}$.

If any check fails, D-RFR reports failure. The subsequent cell decoder also verifies that F and G split into the required distinct linear factors over $\mathcal{U}^*$ before accepting the recovered signed set.

The original RFR subroutine costs $O(\ell \log^2 \ell)$ field operations. The additional operations above—removing a power of Z , truncated series inversion, normalization, gcd computation, and final validation—fit within the same asymptotic bound when standard fast polynomial arithmetic is used. Hence D-RFR runs in $O(\ell \log^2 \ell)$ field operations. In our application, ℓ is a fixed constant, so the reconstruction cost per decoding attempt is constant in the streaming parameters.

D Supplementary Experimental Results

D.1 Prior Works Compared in Section 6.3

We compare XYZ-Sketch against several representative set-reconciliation methods: MiniSketch[4], a high-performance BCH-based algebraic scheme which is an implementation of Pinsketch[8]; CPISync[25], a characteristic-polynomial-based method based on CPI[18]; Rateless IBLT[28], which provides rate-compatible reconciliation; and a C++ implementation of conventional IBLT [1]. All methods are evaluated under the same workload and success target.

D.2 Success Probability Heatmap for a and z

Figure 3 shows representative peeling-success heatmaps for several fixed (d, M) pairs. Each cell reports the measured success rate at a grid point (a, z) , and the yellow box marks the point selected by the fitted heuristic. Across scales, the selected point generally lies within or near a high-success region, while the shape and location of this region vary with d and M .



Fig. 3. Selected fixed- M peeling-success heatmaps. Colors and cell values indicate the measured success rate; yellow boxes mark the points selected by the fitted heuristic.

E Pseudocode for XYZ-Sketch

This appendix gives the detailed pseudocode used by Section 3.

- Algorithm 1 corresponds to the high-capacity cell update;
- Algorithm 2 corresponds to sketch subtraction;
- Algorithm 3 corresponds to local cell decoding and verification;
- Algorithm 4 corresponds to the global ℓ -peeling decoder.

We treat D-RFR as a subroutine: given $R \bmod Z^\ell$ and a signed count m , it returns a monic coprime pair (F, G) or reports failure. The reconstruction procedure is given in Appendix C.

Each cell $\mathcal{B}[i]$ stores three fields:

- (1) A signed count $\mathcal{B}^c[i] \in \mathbb{Z}/(2\ell + 1)\mathbb{Z}$;
- (2) A truncated polynomial field $\mathcal{B}^p[i] \in \mathbb{F}_p[Z]/(Z^\ell)$;
- (3) A fingerprint field $\mathcal{B}^{fp}[i] \in \mathbb{Z}_q$ when fingerprint verification is enabled.

The initial state is $\mathcal{B}^c[i] = 0$, $\mathcal{B}^p[i] = 1$, and $\mathcal{B}^{fp}[i] = 0$ for every cell i . For an element x , let $\Gamma(x) \subseteq [M]$ be the deduplicated support generated by the placement rule in Section 3.3. Since $x \in \mathcal{U}^*$, the factor $Z - x$ is invertible modulo Z^ℓ .

Algorithm 1: XYZ-Sketch: Update Procedure

```

1 Procedure UpdateCell( $i, x, \sigma$ ):
2    $\mathcal{B}^c[i] \leftarrow (\mathcal{B}^c[i] + \sigma) \bmod (2\ell + 1)$ ;
3   if  $\sigma = +1$  then
4      $\mathcal{B}^p[i] \leftarrow \mathcal{B}^p[i] \cdot (Z - x) \bmod Z^\ell$ ;
5     if fingerprint is enabled then
6        $\mathcal{B}^{fp}[i] \leftarrow \mathcal{B}^{fp}[i] + h_{fp}(x)$ ;
7   else
8      $\mathcal{B}^p[i] \leftarrow \mathcal{B}^p[i] / (Z - x) \bmod Z^\ell$ ;
9     if fingerprint is enabled then
10       $\mathcal{B}^{fp}[i] \leftarrow \mathcal{B}^{fp}[i] - h_{fp}(x)$ ;
11 Procedure Update( $x, \sigma$ ):
12   for  $i \in \Gamma(x)$  do
13     UpdateCell( $i, x, \sigma$ );

```

Here $\sigma = +1$ is used for inserting x into the maintained set, and $\sigma = -1$ is used for deleting x from it. If the stream is insertion-only, only the case $\sigma = +1$ is used.

In a valid execution, the queue contains the M initial cell indices and at most k additional indices for each of the at most d recovered elements. Therefore, processing more than $M + kd$ queue entries can only occur after a false-positive decoding path, in which case the decoder safely reports failure.

Algorithm 2: XYZ-Sketch: Subtract Procedure

```

1 Function Subtract( $\mathcal{B}_A, \mathcal{B}_B$ ):
2   Initialize an empty residual sketch  $\mathcal{D}$ ;
3   for  $i \in [M]$  do
4      $\mathcal{D}^c[i] \leftarrow (\mathcal{B}_A^c[i] - \mathcal{B}_B^c[i]) \bmod (2\ell + 1)$ ;
5      $\mathcal{D}^p[i] \leftarrow \mathcal{B}_A^p[i] / \mathcal{B}_B^p[i] \bmod Z^\ell$ ;
6     if fingerprint is enabled then
7        $\mathcal{D}^{fp}[i] \leftarrow \mathcal{B}_A^{fp}[i] - \mathcal{B}_B^{fp}[i]$ ;
8   return  $\mathcal{D}$ ;
  
```

Algorithm 3: XYZ-Sketch: TryDecodeCell

```

1 Function TryDecodeCell( $\mathcal{D}, i$ ):
2    $m \leftarrow \mathcal{D}^c[i]$ ;
3   if  $m > \ell$  then
4      $m \leftarrow m - (2\ell + 1)$ ;
5   if  $|m| > \ell$  then
6     return FAIL;
7    $(F, G) \leftarrow \text{D-RFR}(\mathcal{D}^p[i], m, \ell)$ ;
8   if D-RFR() reports failure then
9     return FAIL;
10   $\Delta_A \leftarrow \text{FindRoot}(F)$ ;
11   $\Delta_B \leftarrow \text{FindRoot}(G)$ ;
12  if  $F \neq \prod_{x \in \Delta_A} (Z - x)$  or  $G \neq \prod_{x \in \Delta_B} (Z - x)$  then
13    return FAIL;
14  if  $|\Delta_A| + |\Delta_B| > \ell$  or  $|\Delta_A| - |\Delta_B| \neq m$  then
15    return FAIL;
16  for  $x \in \Delta_A \cup \Delta_B$  do
17    if  $i \notin \Gamma(x)$  then
18      return FAIL;
19  if fingerprint is enabled then
20    if  $\sum_{x \in \Delta_A} h_{fp}(x) - \sum_{x \in \Delta_B} h_{fp}(x) \neq \mathcal{D}^{fp}[i]$  then
21      return FAIL;
22  return  $(\Delta_A, \Delta_B)$ ;
  
```

Algorithm 4: XYZ-Sketch: Decode Procedure

```

1 Procedure Peel( $x, \tau$ ):
    //  $\tau = +1$  for  $x \in A \setminus B$ , and  $\tau = -1$  for  $x \in B \setminus A$ .
2   for  $j \in \Gamma(x)$  do
3        $\mathcal{D}^c[j] \leftarrow (\mathcal{D}^c[j] - \tau) \bmod (2\ell + 1)$ ;
4       if  $\tau = +1$  then
5            $\mathcal{D}^p[j] \leftarrow \mathcal{D}^p[j] / (Z - x) \bmod Z^\ell$ ;
6           if fingerprint is enabled then
7                $\mathcal{D}^{fp}[j] \leftarrow \mathcal{D}^{fp}[j] - h_{fp}(x)$ ;
8       else
9            $\mathcal{D}^p[j] \leftarrow \mathcal{D}^p[j] \cdot (Z - x) \bmod Z^\ell$ ;
10          if fingerprint is enabled then
11               $\mathcal{D}^{fp}[j] \leftarrow \mathcal{D}^{fp}[j] + h_{fp}(x)$ ;
12       $Q.\text{Push}(j)$ ;

13 Function Decode( $\mathcal{D}$ ):
14      $(S_A, S_B) \leftarrow (\emptyset, \emptyset)$ ;
15     Initialize an empty queue  $Q$ ;
16      $t \leftarrow 0$ ;
17     for  $i \in [M]$  do
18          $Q.\text{Push}(i)$ ;
19     while  $Q$  is not empty do
20          $t \leftarrow t + 1$ ;
21         if  $t > M + kd$  then
22             return FAIL;
23          $i \leftarrow Q.\text{Front}()$ ;
24          $Q.\text{Pop}()$ ;
25          $R \leftarrow \text{TryDecodeCell}(\mathcal{D}, i)$ ;
26         if  $R = \text{FAIL}$  then
27             continue;
28          $(\Delta_A, \Delta_B) \leftarrow R$ ;
29         if  $\Delta_A \cup \Delta_B = \emptyset$  then
30             continue;
31         for  $x \in \Delta_A$  do
32              $S_A \leftarrow S_A \cup \{x\}$ ;
33             Peel( $x, +1$ );
34         for  $x \in \Delta_B$  do
35              $S_B \leftarrow S_B \cup \{x\}$ ;
36             Peel( $x, -1$ );
37     if  $\mathcal{D}^c[i] = 0$  and  $\mathcal{D}^p[i] = 1$  for every  $i$ ,
38     and  $\mathcal{D}^{fp}[i] = 0$  for every  $i$  when fingerprint is enabled then
39         return  $(S_A, S_B)$ ;
40     return FAIL;

```
